Quantum logics as relational monoids

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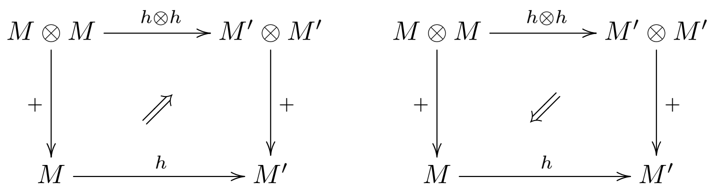
1 Monoids in Rel

It is well known that the category **Rel** of sets and relations, when equipped with the cartesian product of sets (denoted by $\otimes$) forms a symmetric monoidal category. Moreover, **Rel** is a locally posetal (or thin) 2-category, where the 2-cells are given by set inclusion of relations. In the present paper we deal with monoids in **Rel**.

Let $(M, +, 0)$ and $(M', +, 0)$ be monoids in **Rel**. Consider the following pair of diagrams



We say that a relation $h : M \to M$ is a *lax morphism* if $h \circ 0 \subseteq 0$ and the diagram on the left commutes; we say that h is an *oplax morphism* if $0 \subseteq h \circ 0$ and the diagram on the right commutes; h is a *morphism* if it is both lax and oplax.

The aim of this note is to show that several notions in the area of quantum logics can be expressed as the fact that some relation is a lax or an oplax endomorphism of a monoid in **Rel**.

2 Effect algebras as commutative monoids in Rel

An *effect algebra* [2, 5, 4] is a partial algebra $(E; +, 0, 1)$ with a binary partial operation $+$ and two nullary operations $0, 1$ such that $+$ is commutative, associative and the following pair of conditions is satisfied:

(E3) For every $a \in E$ there is a unique $a' \in E$ such that $a+a'$ exists and $a+a' = 1$.
(E4) If $a + 1$ is defined, then $a = 0$.

The $+$ operation is then cancellative and 0 is neutral. Since effect algebras are partial commutative monoids, they are commutative relational monoids. Every effect algebra is a poset under the partial order given by $a \leq b$ iff $(\exists c)\ a = b + c$. An immediate observation shows that this definition comes from the 2-category

Rel: if E is an effect algebra, then $\leq : E \to E$ is a left Kan extension of $+$ along the projection $p_1 : E \otimes E \to E$. Note that $\geq = \leq^{op}$ is an oplax endomorphism of E.

An important subclass of effect algebras are effect algebras with the Riesz decomposition property [6]: if $a \leq b_1 + b_2$, then there are a_1, a_2 such that $a_1 \leq b_1$, $a_2 \leq b_2$ and $a = a_1 + a_2$. It is easy to see that an effect algebra E satisfies this property if and only if $\geq$ is a lax endomorphism of E (so $\geq$ is then an endomorphism).

In [1] Chevalier and Pulmannová considered the following properties of a relation $\sim : P \to P$ on a partial commutative monoid.

(C1) $\sim$ is an equivalence.
(C2) If $a_1 \sim a_2$, $b_1 \sim b_2$, $a_1 + b_1$ exists and $a_2 + b_2$ exists, then $a_1 + b_1 \sim a_2 + b_2$.
(C5) If $c \sim (a_1 + b_1)$ then there are $a_2 \sim a_1$ and $b_2 \sim b_1$ such that $c = a_2 + b_2$.

They proved that for every $\sim$ satisfying these conditions, $P/\sim$ is a partial commutative monoid.

The condition (C5) means exactly that $\sim$ is a lax endomorphism of P and (C2) can be expressed as a 2-cell, as well.

3 Test spaces as commutative semigroups in Rel

A *test space* [3, 7] is a pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a system of subsets of X, called *tests*, such that no two elements of $\mathcal{T}$ are comparable. A subset of a test is an *event*. Two events a, b are said to be *orthogonal* (in symbols $a \perp b$) if they are disjoint and their union is an event. It is obvious that the set of all events of a test space equipped with the disjoint union of orthogonal events $+$ and $\emptyset$ is a partial commutative monoid.

If $a \perp b$ and $a \cup b$ is a test, then they are *complements* of each other (in symbols $a \operatorname{co} b$). We say that events a, b are *perspective* (in symbols $a \sim b$) is they share a complement; note that $\sim = \operatorname{co} \circ \operatorname{co}$ as arrows in **Rel**.

A test space is called *algebraic* if for every triple a_1, a_2, b of events $a_1 \sim a_2$ and $a_1 \operatorname{co} b$ imply $a_2 \operatorname{co} b$; note that this just means that $\operatorname{co} = \operatorname{co} \circ \operatorname{co} \circ \operatorname{co}$ as arrows in **Rel**. In this case, $\sim$ is an equivalence on the set of all events and the equivalence classes of $\sim$ form an *orthoalgebra*, that means, an effect algebra such that if $a + a$ exists, then $a = 0$.

Proposition 1. *Let $(X, \mathcal{T})$ be a test space. Let us write $(A, +, \emptyset)$ for the partial commutative monoid of the events of $(X, \mathcal{T})$. The following are equivalent.*

1. *$(X, \mathcal{T})$ is algebraic.*
2. *$\sim$ is an equivalence relation and an oplax endomorphism of A.*
3. *$\sim$ is a preorder and an oplax endomorphism of A.*
4. *$(A, \tilde{+})$ is associative, where $\tilde{+} = \sim \circ +$.*

We note that, for a general algebraic test space, $(A, \tilde{+})$ is not necessarily a monoid, since the unit axiom need not hold.

Moreover, if $\sim$ is a preorder and $\sim$ is an endomorphism of A, then $A/\sim$ is a Boolean algebra.

These results suggest that the category of commutative monoids in **Rel** forms a suitable framework for studying certain aspects of quantum logics.

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