On maps between Stone-Čech compactifications induced by lattice homomorphisms

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Abstract. Broverman [2] has shown that if X and Y are completely regular Hausdorff spaces, and $t\colon \boldsymbol{Z}(Y) \to \boldsymbol{Z}(X)$ is a lattice homomorphism between the lattices of their zero-sets, then there is a continuous map $\tau\colon \beta X \to \beta Y$ induced by t. In this talk I will expound this idea and supplement Broverman's results by first showing that this phenomenon holds in the category of completely regular frames. Among results I will present, which were not considered by Broverman, are necessary and sufficient conditions (in terms of properties of the map t) for the induced map τ to be (i) the inclusion of a subspace, (ii) surjective, and (iii) irreducible. Time allowing, I will show that if X and Y are pseudocompact, then t pulls back z-ultrafilters to z-ultrafilters if and only if $\mathrm{cl}_{\beta X} t(Z) = \tau^{\leftarrow}[\mathrm{cl}_{\beta Y} Z]$ for every $Z \in \boldsymbol{Z}(Y)$ if and only if t is σ-homomorphism.

Keywords: Tychonoff space, Stone-Čech compactification, zero-set, lattice homomorphism, σ-frame homomorphism, cozero part of a frame

1 Introduction

Let X and Y be Tychonoff spaces and $\boldsymbol{Z}(X)$ and $\boldsymbol{Z}(Y)$ be the lattices of their zero-sets. In [2], Broverman shows that any lattice homomorphism (throughout understood to preserve the bottom and the top elements) $t\colon \boldsymbol{Z}(Y) \to \boldsymbol{Z}(X)$ induces a continuous map $\tau\colon \beta X \to \beta Y$. This is how he does it. Recall that, in the notation of Gillman and Jerison [3], for any $p \in \beta X$, $\boldsymbol{A}^p$ is the z-ultrafilter on X given by

$$Z \in \boldsymbol{A}^p \iff p \in \mathrm{cl}_{\beta X} Z.$$

Now $t^{\leftarrow}[\boldsymbol{A}^p]$ is a prime z-filter in Y, and is therefore contained in some unique z-ultrafilter $\boldsymbol{A}^q$ on Y. Broverman shows that the function $\tau\colon \beta X \to \beta Y$ defined by $\tau(p) = q$ is a continuous map.

Our approach in obtaining such an induced map between Stone-Čech compactifications of frames is different. We take the following categorical path. Let DLat denote the category of bounded distributive lattices and their homomorphisms. Recall that the ideal-lattice functor $\mathfrak{J}\colon \mathsf{DLat} \to \mathsf{Frm}$ sends $A \in \mathsf{DLat}$ to the frame $\mathfrak{J}A$ of ideals of A, and sends a lattice homomorphism $\phi\colon A \to B$ to the frame homomorphism $\mathfrak{J}\phi\colon \mathfrak{J}A \to \mathfrak{J}B$ given by

$$\mathfrak{J}\phi(I) = \{b \in B \mid b \le \phi(a) \text{ for some } a \in I\}.$$

Now let L and M be completely regular frames and $\phi\colon \mathrm{Coz}\, L \to \mathrm{Coz}\, M$ be a lattice homomorphism. Then ϕ preserves the completely below relation, $\prec\!\!\prec$, and hence $\mathfrak{J}\phi(I) \in \beta M$ whenever $I \in \beta L$. Since βL and βM are subframes of $\mathfrak{J}(\mathrm{Coz}\, L)$ and $\mathfrak{J}(\mathrm{Coz}\, M)$ respectively, it follows that the restriction of $\mathfrak{J}\phi$ to βL is a frame homomorphism into βM. We denote it by $\bar{\phi}$. The aim of this note is to explore some properties of the frame homomorphism $\bar{\phi}$ with the view to obtaining new results concerning the map $\tau\colon \beta X \to \beta Y$ induced by a lattice homomorphism $t\colon \boldsymbol{Z}(Y) \to \boldsymbol{Z}(X)$. Sample results are given in the next section.

2 Some main results

We borrow the adjectives "dense" and "codense" from frames and use them to describe lattice homomorphisms in the same way as they describe frame homomorphisms.

Proposition 1. *The homomorphism $\bar{\phi}\colon \beta L \to \beta M$ is one-one iff ϕ is dense.*

Proof (Outline). THe proof hinges on the following description of the right adjoint of $\bar{\phi}$. For any $J \in \beta M$,

$$\begin{aligned}\bar{\phi}_*(J) &= \bigvee\{r_L(c) \mid c \in \mathrm{Coz}\, L \text{ and } \phi(c) \in J\}\\ &= \bigcup\{r_L(c) \mid c \in \mathrm{Coz}\, L \text{ and } \phi(c) \in J\}.\end{aligned}$$

Corollary 1. *Let $t\colon \boldsymbol{Z}(Y) \to \boldsymbol{Z}(X)$ be a lattice homomorphism. The induced map $\tau\colon \beta X \to \beta Y$ is onto iff t is codense.*

Following Ball, Hager and Walters-Wayland [1], we formulate the following definition.

Definition 1. *A lattice homomorphism $\phi\colon \mathrm{Coz}\, L \to \mathrm{Coz}\, M$ is* uplifting *if whenever $u \vee v = 1$ in $\mathrm{Coz}\, M$, then there exist $a, b \in CozL$ such that $a \vee b = 1$, $\phi(a) \le u$ and $\phi(b) \le v$. We say a lattice homomorphism $t\colon \boldsymbol{Z}(Y) \to \boldsymbol{Z}(X)$ is* deflating *if, for any disjoint zero-sets E and F of X, there are disjoint zero-sets G and H of Y such that $E \subseteq t(G)$ and $F \subseteq t(H)$.*

Proposition 2. *The following conditions are equivalent for a lattice homomorphism $\phi\colon \mathrm{Coz}\, L \to \mathrm{Coz}\, M$.*

1. *$\bar{\phi}\colon \beta L \to \beta M$ is onto.*
2. *ϕ is uplifting.*
3. *Whenever $u \prec\!\!\prec v$ in $\mathrm{Coz}\, M$, there are elements $a \prec\!\!\prec b$ in $\mathrm{Coz}L$ such that $u \le \phi(a)$ and $\phi(b) \le v$.*

Recall from [4, Lemma II 2.1] that if $f\colon X \to Y$ is a continuous map between Tychonoff spaces, then $\mathfrak{O}f\colon \mathfrak{O}Y \to \mathfrak{O}X$ is one-one if and only if f is the inclusion of a subspace.

Corollary 2. *Let $t\colon \boldsymbol{Z}(Y) \to \boldsymbol{Z}(X)$ be a lattice homomorphism. The induced map $\tau\colon \beta X \to \beta Y$ is the inclusion of a subspace iff t is deflating.*

A frame homomorphism $h\colon L \to M$ is $*$-*dense* if, for any $a \in L$

$$h_a(a) = 0 \quad \Longrightarrow \quad a = 0.$$

Recall that a surjective continuous map is called *irreducible* if it sends no proper closed subset of its domain onto its codomain. Since, for any continuous map $f\colon X \to Y$, and any $U \in \mathfrak{O}X$,

$$(\mathfrak{O}f)_*(U) = Y \smallsetminus \overline{f[X \smallsetminus U]},$$

it follows that

> *a closed continuous surjection $f\colon X \to Y$ is irreducible iff the frame map $\mathfrak{O}f\colon \mathfrak{O}Y \to \mathfrak{O}X$ is $*$-dense.*

Let us call a lattice homomorphism $\psi\colon A \to B$ *inverse-dense* if, for any ideal J of B, $\psi^{\leftarrow}[J] = \{0\}$ implies $J = \{0\}$.

Proposition 3. *The map $\bar{\phi}\colon \beta L \to \beta M$ is $*$-dense iff ϕ is inverse-dense.*

Corollary 3. *Let $t\colon \boldsymbol{Z}(Y) \to \boldsymbol{Z}(X)$ be a codense lattice homomorphism. The induced map $\tau\colon \beta X \to \beta Y$ is irreducible iff for every nontrivial z-filter $\mathcal{F}$ in X, there is a zero-set $Z \neq Y$ of Y such that $t(Z) \in \mathcal{F}$.*

References

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