Leibniz and Suszko Filters for Non-Protoalgebraic Logics

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Protoalgebraic logics are the logics that have a very weak form of implication, namely a set of formulas $x \Rightarrow y$ in two variables that collectively satisfy modus ponens ($x, x \Rightarrow y \vdash y$) and identity ($\vdash x \Rightarrow x$). In the study of this class of logics, Leibniz filters (introduced in [6]) play a fundamental rôle. They allow to associate with every protoalgebraic logic $\mathcal{S}$, the logic defined by its Leibniz filters, known as the strong version of $\mathcal{S}$. The study of the interplay of the two logics provides some light on the phenomena of pairs of logics strongly related that we find in many areas of non-classical logics. For example, the global modal consequence relation of the class of all Kripke frames is the strong version of the local modal consequence given by that class, and the algebraizable logic of the orthomodular lattices is the strong version of the logic preserving degrees of truth of this variety of algebras. Only very well-behaved protoalgebraic logics coincide with their strong version. This happens with classical and with intuitionistic logic.

Many interesting logics are nevertheless not protoalgebraic. For example positive modal logic, Dunn and Belnap's four-valued logic, the logic preserving degrees of truth from the variety of integral commutative residuated lattices, some subintuitionistic logics, etc. In [1] we have introduced a more general notion of Leibniz filter than the one in [6], as well as the notion of Suszko filter, both applicable for arbitrary logics and, in particular, for the study of the relation of some non-protoalgebraic logics with very close logics that turn out to be the logic of their Leibniz filters.

In the talk we aim to present the new concepts of Leibniz and Suszko filter and to characterize the Leibniz and Suszko filters of the non-protoalgebraic logics previously mentioned, among others, as well as to see the relation between the Leibniz and Suszko filters in those examples. We will also discuss the logic of the Leibniz filters associated with them and see that it coincides with the logic of their Suszko filters. For example, the logic of the Leibniz filters of the logic preserving degrees of truth of the variety of MV-algebras is the Lukasiewicz infinite-valued logic, and the logic of the Leibniz filters of positive modal logic is the positive fragment of the global consequence relation of the class of all Kripke frames.

We will also consider the case of residuated lattices not necessarily integral, as well as that of residuated lattices without a multiplicative constant. Interestingly enough, the algebraizable logic of any of these classes, say $\mathcal{S}$, does not correspond to the logic induced by the Leibniz filters of the logic $\mathcal{S}'$ preserving degrees of truth, as one would expect from the examples mentioned so far. It turns out though, that in both cases there exists a logic $\mathcal{S}''$, as far as we know not previously considered in the literature, in between the logic $\mathcal{S}'$ preserving degrees of truth and $\mathcal{S}$, such that $\mathcal{S}$ is the logic induced by Leibniz filters of this intermediate logic $\mathcal{S}''$.

Let $\mathcal{S}$ be a (sentential) logic. The $\mathcal{S}$-filters of an algebra $\boldsymbol{A}$ of the appropriate type are the subsets of $\boldsymbol{A}$ that are closed under the interpretations of the rules and theorems of $\mathcal{S}$. The set of $\mathcal{S}$-filters is denoted by $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. A congruence θ of an algebra $\boldsymbol{A}$ is *compatible with a set* $F \subseteq A$ when F is a union of equivalence classes of θ. Given $F \subseteq A$, the largest congruence compatible with F always exists, it is denoted by $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ and it is known as the *Leibniz congruence* of F. When considered over the $\mathcal{S}$-filters of $\boldsymbol{A}$, the map $F \mapsto \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ is called the *Leibniz operator on* $\boldsymbol{A}$. By imposing additional conditions over the Leibniz operator (such as order-preserving, injectivity, commuting with inverse images of homomorphisms, etc.), the main classification of logics in abstract algebraic logic is obtained: *the Leibniz hierarchy.* For details, see [2, 7, 5]. For non-protoalgebraic logics a different operator, the Suszko operator ([2, 4]), has proven to be useful. For each algebra $\boldsymbol{A}$, it is the map from the $\mathcal{S}$-filters to the congruences given by $F \mapsto \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$, where $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$ denotes the congruence $\bigcap\{\boldsymbol{\Omega}^{\boldsymbol{A}}(G) : G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}, F \subseteq G\}$, which is known as the *Suszko congruence* of F w.r.t. $\mathcal{S}$. It is the greatest congruence compatible with all the $\mathcal{S}$-filters of $\boldsymbol{A}$ that include F. On protoalgebraic logics the Suszko and the Leibniz operators coincide. By imposing additional conditions over the Suszko operator one can also characterize several classes in the Leibniz hierarchy, as shown in [1].

The notions of Leibniz and Suszko filters for arbitrary logics as introduced in [1] are defined as follows. Let $\mathcal{S}$ be a logic, $\boldsymbol{A}$ an algebra and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. We say that F is a *Leibniz filter* if F is the least element of the class

$$[\![F]\!]^{*} := \big\{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\big\}.$$

Note that for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ the class $[\![F]\!]^{*}$ has always a least element. This least element is always Leibniz. The set of all Leibniz filters of $\boldsymbol{A}$ is denoted by $\mathcal{F}i^{*}_{\mathcal{S}}\boldsymbol{A}$. If the logic $\mathcal{S}$ is protoalgebraic, a filter $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is Leibniz if and only if F is the least element of the class $[F] := \{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\}$. This was the notion of Leibniz filter introduced in [6], and in fact, these classes had already been pointed out in [5] and explicitly defined in [3]. Thus the present concept generalizes the existing one in a way to be applied to any logic, protoalgebraic or not, and we aim to show in the talk its usefulness. In a similar way, we define Suszko filters by considering the Suszko operator instead of the Leibniz one. Let $\mathcal{S}$ be a logic, $\boldsymbol{A}$ an algebra and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. We say that F is a *Suszko filter* if F is the least element of the class

$$[\![F]\!]^{\mathrm{Su}} := \big\{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\big\}.$$

The set of the Suszko filters of $\boldsymbol{A}$ is denoted by $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$.

In [1], the notions of Leibniz and Suszko filters are shown to be closely related to the main classes of logics in the Leibniz hierarchy. For instance, a logic is truth-equational if and only if, for every $\boldsymbol{A}$, every filter of $\boldsymbol{A}$ is a Suszko filter. Also, by restricting the Leibniz operator to the Suszko filters, a new isomorphism theorem for protoalgebraic logics is proved, in the same spirit as the well-known isomorphism theorems for weakly algebraizable ([3]) and algebraizable logics ([8]): A logic $\mathcal{S}$ is protoalgebraic if and only if the Leibniz operator restricted to the Suszko filters $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Co}_{\mathsf{Alg}^*\mathcal{S}}\boldsymbol{A}$ is an order isomorphism, for every $\boldsymbol{A}$. If time permits we will also consider in the talk some of these findings.

References

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