# Topological Correctness Criteria for Linear Logic Proof Structures

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Since its inception in 1987 linear logic (LL, [3]) has changed the proof theoretical way of dealing with proofs and cut elimination. These tasks were traditionally carried out by means of Gentzen sequent calculi with the consequence that the most part of these works were engrossed by tedious commutations of rules. This situation has changed with the accession of the new geometrical syntax for proofs, known as *proof nets*. Proof nets are parallel presentations of sequential proofs of linear logic: they quotient classes of equivalent proofs, modulo irrelevant permutations of inferences. The key ingredients of a proof net syntax are:

- a graphical syntax (i.e., *proof structures*),
- a topological *correctness criterion*, defining proof nets among proof structures,
- an *interpretation of the sequent calculus* syntax,
- a *cut elimination* procedure.

The main properties under consideration for a good notion of proof nets are:

1. *soundness of the interpretation*: the graphs associated with a sequent calculus proof, which are usually given as proof structures, are indeed proof nets;
2. *stability of correctness under cut elimination*;
3. *sequentialization theorem*: each proof net is the image of at least one proof of the sequent calculus;
4. *functional interpretation*: the interpretation of sequent calculus proofs into proof nets is a function;
5. *canonical representation* of proofs of the sequent calculus: proofs equal up to (reasonable) commutations of rules are identified (this leads to the representation of the corresponding free categories with proof nets);
6. *completeness of cut elimination*: for any cut node in a proof net a cut elimination step can be applied;
7. *locality of cut elimination*: a cut elimination step only affects the nodes connected to the cut node it is reducing;
8. *strong normalization and confluence* of the cut elimination procedure;
9. *linear complexity of the correctness checking* w.r.t. the number of nodes.

One usually requires at least the first two properties to hold, otherwise it is really difficult to consider the proposed proof net syntax as a real alternative to the sequent calculus. Concerning proof nets, the Multiplicative fragment of LL (MLL) is the perfect setting: all these conditions are satisfied. A lot of work has been done to extend these properties to the Multiplicative and Additive fragment of LL (MALL). The syntax proposed by J.-Y. Girard in [4] is a very important progress: a new (additive boxes free) syntax for proof nets of MALL where each node of a proof structure is weighted by a nonzero monomial (conjunctions or products) of a Boolean algebra generated by the (different) eigen-weights (variables) associated to the & nodes. Unfortunately Girard's proposal is not as good as for MLL. In particular the "canonical representation" property (point 5) and the "functional interpretation" property (point 4) are lost. The monomial constraint imposed on the additive weights prevents from the natural representation of some sequent calculus proofs. There exist proofs with two possible associated proof nets with no way to discriminate them. This problem has been solved by Hughes and van Glabbeek in [6]. However, an interesting challenge is still to find geometrical (i.e., non inductive) characterizations of proof nets, that is correctness criteria (naively, algorithms) for checking those proof structures which correspond to LL proofs; this is particularly true for MALL proof nets. Our idea is that correctness for MALL proof nets should be formulated as simple as possible, following the *modular spirit* of correctness for MLL proof nets (see [3] and [1]). Following that, our topological correctness criterion (see [8] and [9]) is formulated by means of an algorithm which implements simple graph rewriting rules. We extend, indeed, an initial idea of a retraction correctness criterion for MLL proof nets presented in Danos's Thesis ([2]) and subsequently reformulated as a parsing criterion for MELL proof nets by Guerrini-Masini in [5]. Naively, retraction steps simulate sequentialization steps: each retracted (sub)graph corresponds to a correct (sequentializable) (sub)proof structure. Compared with other existing syntaxes for MALL proof nets, like [4] or [6], our retractile correctness criterion does not rely on any notion of *additive box, slice* or *jump*. This effort should simplify the complexity of checking correctness (point 9).

Concluding, our proof nets syntax satisfies points 1, 2, 3, 6, 7 and 8 above, moreover, it is stable under cut elimination ([7]).

## References

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