Witnessed Models and Skolemization in Substructural Logics

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Skolemization is an important ingredient of automated reasoning methods in (fragments of) first-order classical logic. A sentence $(\forall \bar{x})(\exists y)\varphi(\bar{x}, y)$ is satisfiable if and only if $(\forall \bar{x})\varphi(\bar{x}, f(\bar{x}))$ is satisfiable, where f is a function symbol not occurring in φ. The satisfiability of a sentence in prenex form can therefore be reduced to the satisfiability of a universal sentence; Herbrand's theorem then allows a further reduction to the satisfiability of a set of propositional formulas.

In non-classical logics, the situation is not so clear. Consequence does not reduce to satisfiability, and, in general, sentences must be considered both as premises and as conclusions of consequences. Moreover, not all formulas are equivalent to prenex formulas. Skolemization can then be more carefully defined where strong occurrences of quantifiers in subformulas are replaced on the left, and weak occurrences on the right. However, satisfiability or validity, or, more generally, consequence, may not be preserved. Notably, in intuitionistic logic, formulas such as $\neg\neg(\forall x)P(x) \to (\forall x)\neg\neg P(x)$ do not skolemize (see, e.g., [4]).

For the consequence relation $\models$ of a first-order non-classical logic, the goal of Skolemization may be understood as an algorithmic procedure that takes as input a first-order sentence φ and outputs first-order sentences $sk^l(\varphi)$ and $sk^r(\varphi)$ containing, respectively, no strong or weak quantifier occurrences and satisfying for any set of formulas $\Sigma \cup \{\psi\}$,

$$\Sigma \cup \{\varphi\} \models \psi \quad \Leftrightarrow \quad \Sigma \cup \{sk^l(\varphi)\} \models \psi$$
$$\Sigma \models \varphi \quad \Leftrightarrow \quad \Sigma \models sk^r(\varphi).$$

This goal typically fails for first-order non-classical logics if restricted to the Skolemization process mentioned above, an interesting exception being first-order Łukasiewicz logic (see, e.g., [2]). However, allowing multiple new function symbols in the Skolemization step broadens the applicability of the approach considerably. In particular, this "parallel Skolemization" method has been used by Baaz and Iemhoff [1] to obtain Skolemization results for first-order intermediate logics whose Kripke models (with and without the constant domains condition) admit the finite model property. In the work reported here, we extend parallel Skolemization to the setting of first-order substructural logics.

Let $\mathbb{K}$ be a class of complete $\mathrm{FL_e}$-algebras (see [5] for details) and let $\models_{\mathbb{K}}$ be the corresponding consequence relation of the first-order substructural logic based on this class, as described in, e.g., [4]. The idea of parallel Skolemization is to replace certain subformulas $(Qx)\psi(x,\bar{y})$ with

$$\psi' = \begin{cases} \bigvee_{i=1}^{n} \psi(f_i(\bar{y}), \bar{y}) & \text{if } Q = \exists \\ \bigwedge_{i=1}^{n} \psi(f_i(\bar{y}), \bar{y}) & \text{if } Q = \forall \end{cases}$$

for some fixed n, where $f_1, \ldots, f_n$ are new function symbols. For a sentence φ, let $sk_n^l(\varphi)$ and $sk_n^r(\varphi)$ be the results of applying this process repeatedly to leftmost strong or weak, respectively, occurrences of quantifiers in φ. We say that $\models_{\mathbb{K}}$ admits *parallel Skolemization right of degree* n for a sentence φ if for any set of sentences Σ,

$$\Sigma \models_{\mathbb{K}} \varphi \quad \Leftrightarrow \quad \Sigma \models_{\mathbb{K}} sk_n^r(\varphi).$$

Similarly, we say that $\models_{\mathbb{K}}$ admits *parallel Skolemization left of degree* n for φ if for any set of sentences $\Sigma \cup \{\psi\}$,

$$\Sigma \cup \{\varphi\} \models_{\mathbb{K}} \psi \quad \Leftrightarrow \quad \Sigma \cup \{sk_n^l(\varphi)\} \models_{\mathbb{K}} \psi.$$

A characterization for logics $\models_{\mathbb{K}}$ that admit parallel Skolemization of degree n for any sentence is obtained via a generalization of the notion of *witnessed models* (see, e.g., [3]). Roughly, we call a structure based on some algebra $\boldsymbol{A}$ from $\mathbb{K}$ *n-witnessed* if any meet or join of a set of elements from A that could occur in the interpretation of a formula is actually the meet or join, respectively, of n elements from that set. We then say that the consequence relation $\models_{\mathbb{K}}$ has *the n-witnessed model property* if for any set of sentences $\Sigma \cup \{\varphi\}$,

$$\Sigma \models_{\mathbb{K}} \varphi \quad \Leftrightarrow \quad \text{every } n\text{-witnessed model } \mathfrak{M} \text{ of } \Sigma \text{ is a model of } \varphi.$$

In particular, if $\mathbb{K}$ is any finite class of finite FL_e-algebras, then $\models_{\mathbb{K}}$ has the n-witnessed model property for some $n \in \mathbb{N}$, while if $\mathbb{K}$ consists of the standard Łukasiewicz algebra on $[0,1]$, then $\models_{\mathbb{K}}$ has the 1-witnessed model property.

Theorem 1. *If $\models_{\mathbb{K}}$ has the n-witnessed model property, then $\models_{\mathbb{K}}$ admits parallel Skolemization left and right of degree n for all sentences. Moreover, if $\models_{\mathbb{K}}$ is finitary, then the converse implication also holds.*

References

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