Monadic second order logic on infinite words is the model companion of linear temporal logic

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Monadic second order logic and linear temporal logic are two logical formalisms that can be used to describe classes of infinite words, i.e., first-order models based on the natural numbers with order, successor, and finitely many unary predicate symbols.

Monadic second order logic over infinite words (S1S) can alternatively be described as a first-order logic interpreted in $\mathcal{P}(\omega)$, the power set Boolean algebra of the natural numbers, equipped with modal operators for 'initial', 'next' and 'future' states. We prove that the first-order theory of this structure is the model companion of a class of algebras corresponding to the appropriate version of linear temporal logic (LTL).

The proof makes crucial use of two classical, non-trivial results from the literature, namely the completeness of LTL with respect to the natural numbers, and the correspondence between S1S-formulas and Büchi automata.

Let us describe our main result in some more detail. We define an LTL_I-*algebra* to be a tuple $(A, \vee, \neg, \bot, \Diamond, \mathbf{X}, I)$, where $(A, \vee, \neg, \bot)$ is a Boolean algebra, $\Diamond$ is a unary normal modal operator on A, $\mathbf{X}$ is a Boolean endomorphism of A, I is an element of $A \setminus \{\bot\}$, and, for any $a \in A$:

1. $\Diamond a = a \vee \mathbf{X}\Diamond a$,
2. if $\mathbf{X}a \leq a$ then $\Diamond a \leq a$,
3. if $a \neq \bot$ then $I \leq \Diamond a$.
4. $\mathbf{X}I = \bot$.

The class of LTL_I-algebras (which is a *universal* class) algebraizes a version of linear temporal logic without the until connective and with an 'initial element' constant I. An important example of an LTL_I-algebra is the *power set algebra of the natural numbers*, $\mathcal{P}(\omega)$, equipped with the usual Boolean operations, $\Diamond S := \{n \in \omega \mid n \leq s \text{ for some } s \in S\}$, $\mathbf{X}S := \{n \in \omega \mid n + 1 \in S\}$, and $I := \{0\}$. In particular, note that first-order formulas in the signature of LTL_I-algebras, interpreted in $\mathcal{P}(\omega)$, are interdefinable with formulas in the system S1S, monadic second order logic over the natural numbers with order and successor relations.

If T and T^* are first-order theories in the same signature, recall that T^* is called a *model companion* of T if (i) the theories T and T^* prove the same quantifier-free formulas and (ii) any first-order formula is T^*-provably equivalent to a universal formula (i.e., T^* is *model complete*). The model companion of T is unique if it exists, and in this case it is the theory of the existentially complete T-structures [4]. Our main theorem is the following.

Theorem 1. *The first-order theory of the* LTL_I*-algebra* $\mathcal{P}(\omega)$ *is the model companion of the first-order theory of* LTL_I*-algebras.*

To prove Theorem 1, we need to verify (i) and (ii) in the definition of model companion. We give some details about the proofs of these properties.

(i). It suffices to prove that any quantifier-free formula which is valid in $\mathcal{P}(\omega)$ is valid in any LTL_I-algebra. Any quantifier-free formula can be rewritten into an equation of the form $t = \top$, using the equivalence $a \neq \top \iff I \leq \Diamond\neg a$, which is valid in all LTL_I-algebras, and standard facts about Boolean algebras. Item (i) will now follow from the following completeness theorem for LTL_I.

Theorem 2. *If* t *is an* LTL_I*-term and* $\mathcal{P}(\omega) \models t = \top$*, then, for any* LTL_I*-algebra* A*,* $A \models t = \top$*.*

To prove this theorem, we use the following convenient representation for LTL_I-algebras. We define an LTL_I*-space*[1] to be a tuple $(X, \leq, f, x_0)$, where X is a Boolean topological space, $\leq$ is a topological quasiorder on X (i.e., $\uparrow x$ is closed for any $x \in X$ and $\downarrow K$ is clopen for any clopen $K \subseteq X$), $f : X \to X$ is a continuous function, $x_0 \in X$ is a point such that $\{x_0\}$ is clopen, and, for any $x, y \in X$ and clopen $K \subseteq X$:

1. $x \leq f(x)$, and if $x < y$ then $f(x) \leq y$,
2. if $f(K) \subseteq K$ then $\uparrow K \subseteq K$,
3. for any $x \in X$, $x_0 \leq x$,
4. $f^{-1}(\{x_0\}) = \emptyset$.

The *dual algebra* of an LTL_I-space $(X, \leq, f, x_0)$ is the Boolean algebra of clopens of X, equipped with the operations $\Diamond K := \downarrow K$, $\mathbf{X}K := f^{-1}(K)$ and $I := \{x_0\}$. We obtain the following adaptation of the standard Stone-Jónsson-Tarski representation theorem.

Theorem 3. *Any* LTL_I*-algebra is isomorphic to the dual algebra of a unique* LTL_I*-space.*

To prove Theorem 2, we combine Theorem 3 with an adaptation of a filtration argument for LTL [2]. This concludes the proof of item (i).

(ii). We make use of a classical result by Büchi [1], referring to, e.g., [3] for background and more details. For a Büchi automaton A and a formula ψ in S1S, we denote by $L(A)$ the set of infinite words accepted by A, and by $L(\psi)$ the set of infinite words validating ψ.

Theorem 4. *For any formula* ψ *in S1S, there exists a Büchi automaton* A_ψ *such that* $L(A_\psi) = L(\psi)$*. Conversely, for any Büchi automaton* A *there exists an existential S1S-formula* χ_A *such that* $L(\chi_A) = L(A)$*.*

[1] Note that our definition of LTL_I-spaces makes crucial use of the second-order structure (topology) on the underlying Kripke frames. This is necessarily so: the class of LTL_I-algebras is not canonical, so it can not be dual to an elementary class of Kripke frames, by a theorem of Fine.

Towards proving (ii), let $\varphi(v_1, \ldots, v_n)$ be a first-order formula in the signature of LTL_I-algebras. In $\mathcal{P}(\omega)$, φ is equivalent to a formula ψ in the system S1S over the alphabet $\Sigma := \mathcal{P}(v_1, \ldots, v_n)$. Let $A_{\neg\psi}$ be the Büchi automaton for the formula $\neg\psi$, and let $\chi_{A_{\neg\psi}}$ be the existential formula describing this automaton. Then $\psi' := \neg\chi_{A_{\neg\psi}}$ is a universal S1S-formula for which $L(\psi') = L(\psi)$. Now ψ' is equivalent in $\mathcal{P}(\omega)$ to a universal formula φ' in the signature of LTL_I-algebras. The formula φ' is the required universal formula which is equivalent to φ over T^*.

References

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