Admissible Bases via Stable Canonical Rules

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Abstract. We establish the dichotomy property of [7] for multi-conclusion stable canonical rules of [1]. This yields an alternative proof of existence of bases of admissible rules for such well-known systems as **IPC**, **S4**, and **K4**.

1 Introduction

An inference rule is *admissible* in a given logical system L if no new theorems are derived by adding the rule to the rules of inference of L. Whether or not a given rule of inference is admissible in such well-known systems as **IPC**, **S4**, and **K4** was first solved by Rybakov (see the comprehensive book [9] and the references therein). An alternative solution via projectivity and unification was supplied in [3, 4]. Bases for admissible rules were built in [8, 10, 5, 6]. Recently E. Jeřábek [7] developed a new technique for building bases of admissible rules by generalizing Zakharyaschev's canonical formulas [11] to multi-conclusion canonical rules, and developing the *dichotomy property* of canonical rules. This property says that a canonical multi-conclusion rule is either admissible or equivalent to an assumption-free rule. Our goal is to establish the same property for *stable* multi-conclusion canonical rules for **IPC**, **S4**, and **K4**. These rules were recently introduced in [1], where it was shown that each normal modal multi-conclusion consequence relation is axiomatizable by stable multi-conclusion canonical rules. The same result for intuitionistic multi-conclusion consequence relations was established in [2]. The proof methodology we follow is similar to [7] and goes through a semantic characterization of non-admissible stable canonical rules in terms of the finite domains they are built from. In spite of the similarities, the semantic characterization we obtain is rather different than the one given in [7]. For space reasons, we outline our arguments in the case of **K4** only. Full details will be supplied in a forthcoming article.

2 Closed domain condition and stable canonical rules for K4

Definition 1. *Let* $\mathfrak{A} = (A, \Box)$ *and* $\mathfrak{B} = (B, \Box)$ *be* **K4***-algebras, and let* $h : A \to B$ *be a map. We call* h *a* stable homomorphism *if it is a Boolean homomorphism satisfying* $h(\Box a) \leqslant \Box(ha)$ *for each* $a \in A$.

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Definition 2. *Let $\mathfrak{A}, \mathfrak{B}$ be* **K4**-*algebras and let $h : A \to B$ be a stable homomorphism. We say that h satisfies* the closed domain condition (CDC) for $a \in A$ *if $h(\Box a) = \Box h(a)$. We say that h satisfies* the closed domain condition (CDC) for $D \subseteq A$ *if h satisfies* (CDC) *for each $a \in D$.*

We next describe these concepts dually, in terms of descriptive **K4**-frames.

Definition 3. *Let $\mathfrak{F} = (W, R)$ and $\mathfrak{G} = (V, R)$ be descriptive* **K4**-*frames, and let $f : V \to W$ be a continuous map. We call f* stable *if wRv implies $f(w)Rf(v)$ for each $w, v \in V$.*

Definition 4. *Let $\mathfrak{F} = (W, R)$ and $\mathfrak{G} = (V, R)$ be descriptive* **K4**-*frames, and let $f : V \to W$ be a stable map. If U is a clopen subset of W, then we say that f satisfies the* closed domain condition (CDC) for U *if $U \cap R[f(v)] \neq \varnothing \Rightarrow U \cap f(R[v]) \neq \varnothing$ for each $v \in V$. If $\mathfrak{D}$ is a collection of clopen subsets of W, then we say that f satisfies the* closed domain condition (CDC) for $\mathfrak{D}$ *if f satisfies* (CDC) *for each $U \in \mathfrak{D}$.*

Theorem 1. *Let $\mathfrak{A}, \mathfrak{B}$ be* **K4**-*algebras and let $\mathfrak{F}, \mathfrak{G}$ be their dual descriptive* **K4**-*frames. Let $h : A \to B$ be a Boolean homomorphism and let $f : V \to W$ be its dual continuous map. Then h is stable iff f is stable. Moreover, if $D \subseteq A$ and $\mathfrak{D}$ is the corresponding collection of clopen subsets of W, then h satisfies* (CDC) *for D iff f satisfies* (CDC) *for $\mathfrak{D}$.*

Definition 5. *Let $\mathfrak{A}$ be a finite* **K4**-*algebra and let $D \subseteq A$. For each $a \in A$ we introduce a new propositional letter p_a and define the* stable canonical rule *$\rho(\mathfrak{A}, D)$ associated with $\mathfrak{A}$ and D as Γ/Δ, where $\Delta = \{p_a : a \in A, a \neq 1\}$ and $\Gamma = \{p_{a\wedge b} \leftrightarrow p_a \wedge p_b, p_{\neg a} \leftrightarrow \neg p_a, p_{\Box a} \to \Box p_a : a, b \in A\} \cup \{\Box p_a \leftrightarrow p_{\Box a} : a \in D\}$.*

Theorem 2 ([1]). *Let $\mathfrak{A}$ be a finite* **K4**-*algebra, $D \subseteq A$, and $\mathfrak{B}$ be a* **K4**-*algebra. Then $\mathfrak{B} \not\models \rho(\mathfrak{A}, D)$ iff there is a stable embedding $h : A \rightarrowtail B$ satisfying* (CDC) *for D. Consequently, if $\mathfrak{F}$ is the dual of $\mathfrak{A}$, $\mathfrak{G}$ is the dual of $\mathfrak{B}$, and $\mathfrak{D}$ is the dual of D, then $\mathfrak{B} \not\models \rho(\mathfrak{A}, D)$ iff there is a stable onto map $f : V \to W$ satisfying* (CDC) *for $\mathfrak{D}$.*

Because of this, if $\mathfrak{A} = (A, \Box)$ is a finite **K4**-algebra and $\mathfrak{F} = (W, R)$ is its dual finite **K4**-frame, then we denote the stable canonical rule $\rho(\mathfrak{A}, D)$ by $\rho(\mathfrak{F}, \mathfrak{D})$, where $D \subseteq A$ and $\mathfrak{D} \subseteq \mathcal{P}(W)$ is its dual.

For a formula φ, let $\Box^+\varphi := \varphi \wedge \Box\varphi$. We let $(S^m_{n,\ell})$ be the rule

$$\frac{\bigwedge_{l=1}^{\ell}(\Box x_l \to x_l) \wedge \bigwedge_{k=1}^{m}\Box(r_k \to \Box(r_k \vee \Box^+ q)) \to \bigvee_{i=1}^{n}\Box p_i}{\Box^+ q \to p_1 | \ldots | \Box^+ q \to p_n} \tag{1}$$

and (T_m) be the rule

$$\frac{\bigwedge_{k=1}^{m}(\Diamond r_k \to \Diamond(r_k \wedge \Box^+ q)) \to \bigvee_{i=1}^{n}\Box p_i}{\Box^+ q \to p_1 | \ldots | \Box^+ q \to p_n} \tag{2}$$

Theorem 3. *The rules* $(S^m_{n,\ell})$ *are admissible for all* $n, m, \ell \in \omega$*, and the rules* (T_m) *are admissible for all* $m \in \omega$*.*

Let R^+ be the reflexive closure of R.

Definition 6. *A stable canonical rule* $\rho(\mathfrak{F}, \mathfrak{D})$ *is called* trivial$^\circ$ *if for every* $S \subseteq W$*, there is a reflexive* $w^\circ \in W$ *such that* **(1)** $S \subseteq R[w^\circ]$*; and* **(2)** *For all* $U \in \mathfrak{D}$*, if* $U \cap R[w^\circ] \neq \varnothing$*, then* $U \cap (\{w^\circ\} \cup R^+[S]) \neq \varnothing$*.*

A stable canonical rule $\rho(\mathfrak{F}, \mathfrak{D})$ *is called* trivial$^\bullet$ *if for every* $S \subseteq W$*, there is* $w^\bullet \in W$ *such that* **(3)** $S \subseteq R[w^\bullet]$*; and* **(4)** *For all* $U \in \mathfrak{D}$*, if* $U \cap R[w^\bullet] \neq \varnothing$*, then* $U \cap R^+[S] \neq \varnothing$*.*

A stable canonical rule is trivial *if it is both trivial*$^\circ$ *and trivial*$^\bullet$*.*

Notice that the points x° and $x^\bullet$ can coincide. The dichotomy property mentioned in the introduction can now be stated as follows.

Theorem 4. *The following are equivalent:*

1. $\rho(\mathfrak{F}, \mathfrak{D})$ *is admissible.*
2. $\rho(\mathfrak{F}, \mathfrak{D})$ *is derivable from* $\{S^m_{n,\ell} : m, n, \ell \in \omega\} \cup \{T_m : m \in \omega\}$*.*
3. $\rho(\mathfrak{F}, \mathfrak{D})$ *is not trivial.*
4. $\rho(\mathfrak{F}, \mathfrak{D})$ *is not equivalent to an assumption-free rule.*

Corollary 1. *The rules* $\{S^m_{n,\ell} : m, n \in \omega\} \cup \{T_m : m \in \omega\}$ *form an admissible basis for* **K4***.*

The admissible basis $\{S^m_{n,\ell} : m, n \in \omega\} \cup \{T_m : m \in \omega\}$ is equivalent to the admissible basis for **K4** given in [7].

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