Characterization of Finite Embeddability Property for (Distributive) Residuated Lattices via Regular (Tree) Languages

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A variety V of residuated lattices has the finite embeddability property (shortly FEP) if every finite partial subalgebra $\mathbf{B}$ of $\mathbf{A} \in \mathsf{V}$ is embeddable into a finite member of V. This property is equivalent to the finite model property for the universal theory of V. Consequently, if V is finitely axiomatizable and has the FEP then its universal theory is decidable.

There exists already a bunch of results on the FEP for particular varieties of residuated lattices usually using the same construction due to Blok and van Alten [1]. Nevertheless, it is not clear whether the FEP for a variety of residuated lattices V implies that this construction works for V. In this talk we show that for varieties axiomatized over $\{\vee, \cdot, 1\}$ this is really the case if we generalize the construction a little bit. A similar result holds also for varieties of distributive residuated lattices axiomatized over $\{\wedge, \vee, \cdot, 1\}$ if we use the construction announced in [4].

We formulate the above result more precisely. Recall that a *residuated frame* $\mathbf{W} = \langle \mathbf{M}, Z, N \rangle$ is a triple, where $\mathbf{M}$ is a monoid, Z a set and $N \subseteq M \times Z$ a nuclear relation (see [5]). The nuclear relation N induces a nucleus γ on $\mathcal{P}(\mathbf{M})$ whose basis consists of the sets $\{x \in M \mid x \; N \; z\}$ for $z \in Z$. We call these sets *basic closed sets* of $\mathbf{W}$. The *dual algebra* $\mathbf{W}^+$ of $\mathbf{W}$ is defined as the residuated lattice $\mathcal{P}(\mathbf{M})_\gamma$. A *Gentzen residuated frame* is a tuple $\langle \mathbf{W}, \mathbf{B} \rangle$, where $\mathbf{W}$ is a residuated frame and $\mathbf{B}$ a partial algebra in the signature of residuated lattices such that B generates (as a monoid) $\mathbf{M}$, there is an injection of B into Z, and the relation N satisfies the rules from the the full Lambek calculus (see [5]). We call $\langle \mathbf{W}, \mathbf{B} \rangle$ *antisymmetric* if for all $a, b \in B$ we have $a \; N \; b$, $b \; N \; a$ implies $a = b$.

Theorem 1. *Let V be a variety of residuated lattices axiomatized over $\{\vee, \cdot, 1\}$. Then the following are equivalent:*

1. *V has the FEP.*
2. *For every algebra $\mathbf{A} \in \mathsf{V}$ and a finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ there is an antisymmetric Gentzen residuated frame $\langle \mathbf{W}, \mathbf{B} \rangle$ such that $\mathbf{W}^+ \in \mathsf{V}$ and every basic closed set of $\mathbf{W}$ forms a regular language.*

Using the above theorem, one can reprove most of the known positive results on the FEP via a regularity condition from the language theory. Let V be a

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variety of residuated lattices axiomatized over $\{\vee, \cdot, 1\}$. Given $\mathbf{A} \in \mathsf{V}$ and a finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$, there is always an antisymmetric Gentzen residuated frame $\langle \mathbf{W}, \mathbf{B} \rangle$ such that $\mathbf{W}^+ \in \mathsf{V}$. Namely, $\mathbf{W} = \langle \mathbf{B}^*, B^* \times B^* \times B, N \rangle$, where $\mathbf{B}^*$ is the free monoid generated by B and $x\ N\ \langle u, v, b \rangle$ iff $id^*(uxv) \leq^{\mathbf{A}} b$. The function $id^*\colon B^* \to A$ is the free extension of the inclusion of B into A.

Thus if we want to prove the FEP for the variety V, it suffices to show that the basic closed sets of $\mathbf{W}$ are regular languages. This is easy if V is integral using well-known Higman's lemma and the generalized Myhill-theorem [3].

Theorem 2 ([5]). *Every variety* V *of integral residuated lattices axiomatized over* $\{\vee, \cdot, 1\}$ *has the FEP.*

Similarly, using the fact that a language is regular iff it is permutable and quasi-periodic or co-quasi-periodic (see [7]), one can immediately prove the next two theorems.

Theorem 3 ([9]). *Let* V *be a variety of commutative residuated lattices axiomatized over* $\{\vee, \cdot, 1\}$ *and satisfying* $x^m \leq x^n$ *for* $m \neq n$. *Then* V *has the FEP.*

Theorem 4 ([2]). *Let* V *be a variety of residuated lattices axiomatized over* $\{\vee, \cdot, 1\}$ *satisfying* $xyx = x^2y$ *and* $x^m \leq x^n$ *for* $m \neq n$. *Then* V *has the FEP.*

An analogous characterization of the FEP as in Theorem 1 can be obtained for varieties of distributive residuated lattices if we replace the regular languages by regular tree languages, i.e., sets of terms over $\{\wedge, \cdot\}$ recognizable by a finite tree automaton (for details on regular tree languages see [6]).

Then one can prove the following theorem immediately using Kruskal tree theorem and the generalized Myhill-theorem for tree languages [8].

Theorem 5 ([4]). *Every subvariety of distributive integral residuated lattices axiomatized over* $\{\wedge, \vee, \cdot, 1\}$ *has the FEP.*

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