On One Embedding of Heyting Algebras

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1 Enriched elements of Heyting algebra

Given a Heyting algebra $\mathfrak{A}$, we say that an element $a \in \mathfrak{A}$ is ***enriched*** (in $\mathfrak{A}$) by an element $b \in \mathfrak{A}$ if the following conditions are satisfied:

(a) $a \leq b$,
(b) $b \to a = a$,
(c) $b \leq x \vee (x \to a)$, for any $x \in \mathfrak{A}$.

It is well known (see, e.g., [5]) that, if an element of a Heyting algebra is enrichable, it can be enriched by only one element, that is to say, if a is enriched by b and by c, then $b = c$. A Heyting algebra is ***enrichable*** if each element of it is enrichable. Thus, in an enrichable Heyting algebra $\mathfrak{A}$ modality $\Box$ can be defined in a unique way so that the following identities are satisfied:

(d) $x \leq \Box x$,
(e) $\Box x \to x = x$.
(f) $\Box x \leq y \vee (y \to x)$.

This expansion $\langle \mathfrak{A}, \Box \rangle$ is known as a KM-algebra. KM-algebras are algebraic semantics for KM logic (alias proof-intuitionistic logic) which was introduce by A. V. Kuznetsov; cf. [1, **?**]. This logic has been in focus for the last two decades because of the following theorem: For any $\Box$-free formulas A and B,

$$\mathbf{KM} + A \vdash B \Longleftrightarrow \mathbf{Int} + A \vdash B, \qquad (\textit{Kuznetsov's Theorem})$$

where **Int** is intuitionistic propositional calculus. (This theorem was one of the two key properties used to prove the main theorem of [2], which connected the lattices of the normal extensions of the logics **Int**, **Grz**, **GL** and **KM** in a commutative diagram.) One of the consequences of Kuznetsov's Theorem is the following proposition.

Proposition 1 ([1], Corollary 2). *Any Heyting algebra is embedded into an enrichable Heyting algebra so that both algebras generate one and the same variety.*

In fact, as it was pointed out by the author to Kuznetsov (see [1]), Proposition 1 is equivalent to Kuznetsov's Theorem. (A proof, using model-theory argument, can be found in [4], Section 3.3, Remark 3, in English, and elsewhere and earlier in Russian.) However, a direct (independent from Kuznetsov's Theorem and algebraic) proof of Proposition 1 has not been available. Such a proof is obtained in Theorem 3.

2 Localization of enrichment and embedding

Our goal is twofold: 1) given a Heyting algebra $\mathfrak{A}$, we define such an enrichable $\mathfrak{B}$ that $\mathfrak{A}$ is embedded into $\mathfrak{B}$; and, then, 2) we prove that $\mathfrak{A}$ and $\mathfrak{B}$ generate one and the same variety. However, at first we embed $\mathfrak{A}$ into such $\mathfrak{B}'$, where at least one element of $\mathfrak{A}$ is enrichable. To designate an element in $\mathfrak{A}$, which we are going to enrich, we define a ***τ-expansion*** of a Heyting algebra $\mathfrak{A}$ by adding a nullary operation τ to the signature of $\mathfrak{A}$. Let $\langle \mathfrak{A}, \tau \rangle$ and $\langle \mathfrak{B}, \tau \rangle$ be τ-expansions. We call the latter a ***τ-embrace*** of the former if $\langle \mathfrak{A}, \tau \rangle$ is a subalgebra (up to isomorphism) of $\langle \mathfrak{B}, \tau \rangle$, τ is enriched in $\mathfrak{B}$ by some $b \in \mathfrak{B}$, and $\mathfrak{B}$ is generated by the set $|\mathfrak{A}| \cup \{b\}$.

Theorem 1. *Given a τ-expansion $(\mathfrak{A}, \tau)$, there is its τ-embrace. Moreover, all τ-embraces of $(\mathfrak{A}, \tau)$ are isomorphic to one another.*

Theorem 1 is used to prove the following.

Theorem 2. *Let $\langle \mathfrak{B}, \tau \rangle$ be a τ-embrace of $\langle \mathfrak{A}, \tau \rangle$. Then the Heyting reducts $\mathfrak{B}$ and $\mathfrak{A}$ generate one and the same variety.*

We recall that [3] shows that given a Heyting algebra $\mathfrak{A}$, there is an enrichable Heyting algebra $\boldsymbol{\mathfrak{A}}$ (a direct limit of a infinite family of algebras generated from $\mathfrak{A}$) such that $\mathfrak{A}$ is embedded into $\boldsymbol{\mathfrak{A}}$. With help of Theorems 1 and 2, we obtain the following.

Theorem 3. *Given a Heyting algebra $\mathfrak{A}$, the algebras $\mathfrak{A}$ and $\boldsymbol{\mathfrak{A}}$ generate one and the same variety.*

References

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