The logic of quasi true

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Abstract. We introduce the new logic CL, which is an extension of the infinitely valued Łukasiewicz logic Ł, the language of which enriched by 0-ary connective c that is interpreted as "quasi-false", the algebraic counterpart of which are algebras from a quasi-variety generated by the perfect Chang MV-algebras C signature of which enriched by the constant element **c**. For this aim we introduce a new class **CL** of algebras which is a quasi-variety and the algebras from this quasi-variety we name CL-algebras. Adding a new inference rule to the logic CL, thereby increased a deducibility power, we introduce the logic CL^+ and defining the notion of quasi-true (q-true) formulas it is proved the completeness theorem for this logic.

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1 Introduction

MV-algebras are the algebraic counterpart of Łukasiewicz sentential calculus Ł [1]. Since there are non simple linearly ordered MV-algebras, in this case, infinitesimal elements of an MV-algebra are allowed to be truth values. We look for a logic which is an extension of Ł that is evaluated over a non simple MV-chain. There are MV-algebras which are not semisimple, i.e. the intersection of their maximal ideals (the radical of A) is different from $\{0\}$. Non-zero elements from the radical of A are called infinitesimals. Perfect MV-algebras are those non semisimple MV-algebras generated by their infinitesimal elements. The perfect algebra C is introduced by Chang in [1] which is generated by the one infinitesimal element **c**. Considering the unit interval MV-algebra $[0,1]$ as the structure over which to evaluate a formula of a sentential calculus, one has many possibilities to evaluate a formula. Let us start with Łukasiewicz logic Ł, and evaluate formulasover $[0,1]$. A truth value $x \in [0,1]$, $x \neq 1$, can be considered as the value of a not-true formula. The distance of x from 1 can be considered as expressing how much x is close to be true. Starting from x, assuming $v(\alpha) = x$, where $v(\alpha)$ is the valuation of the formula a, then after a finite number of steps made by the strong disjunction $\oplus$, like $v(\alpha), v(\alpha \oplus \alpha), v(\alpha \oplus \alpha \oplus \alpha), ...$, we get, for every evaluation v, $v(\alpha \oplus ... \oplus \alpha) = 1$, and, similarly using the strong conjunction $\odot$, also $v(\alpha \odot ... \odot \alpha) = 0$. This cannot be for Łukasiewicz tautologies α, indeed we have $v(\alpha) = 1$ and $v(\alpha \odot ... \odot \alpha) = 1$, for all evaluations v. All that is due to the simplicity of $[0,1]$. Assume now to evaluate Ł over a ultrapower $*[0,1]$ of $[0,1]$. Assume there is a formula α such that $v(\alpha)$ is infinitesimally close to 1. We are interested in considering such formulas having a *co-infinitesimal* value for every evaluation. For any of such formulas the behavior has to be intermediate between the behavior of tautologies and the behavior of formulas evaluated into a real number in $[0,1]$. It is reasonable to look at such formulas as *quasi true*. So it is an interesting task to explore the way to make *formal* such a concept of *quasi truth* and to develop logics allowing a generalization of the concept of truth. We look for logics which are extensions of Ł having evaluation over a non simple MV-chain. So, we can think of the logic CL of the perfect MV-algebra C, roughly speaking, as a variant of logic of the concept of "quasi-true", or better of the concept of "infinitesimally close to the truth".

2 CL-algebras

An algebra $A = (A, 0, \mathbf{c}, \neg, \oplus)$ is said to be CL-algebra if $A = (A, 0, \neg, \oplus)$ is MV-algebra and in addition it satisfy the following axioms:

1) $2(x^2) = (2x)^2$,
2) $2\mathbf{c} \odot \neg\mathbf{c} = \mathbf{c}$,
3) $(\mathbf{c} \odot \neg x) \wedge x = 0$,
4) $\mathbf{c} \to \neg\mathbf{c} = 1$,
5) $y^2 \oplus ((2\neg y) \odot (y \odot \neg x)) = 0 \Rightarrow (2x = 1)$,
6) $x \wedge \mathbf{c} = 0 \Rightarrow x = 0$.

Comment. The identity 1) says that a CL-algebra is a member of the subvariety $V(C)$ (= the variety generated by all perfect MV-algebras) of the variety of all MV-algebras. The second 2) says that $\mathbf{c} \neq \mathbf{c} \vee \neg\mathbf{c}$ and $\mathbf{c} \neq 1$. 3) says that $\mathbf{c}$ is the atom in a totally ordered CL-algebra. 4) says that $\mathbf{c} \leq \neg\mathbf{c}$. 5) gives that we exclude the algebras $C_m = \Gamma(Z \times_{lex} \cdots \times_{lex} Z, (1, 0, ..., 0))$ where the number of factors Z is equal to $m \geq 3$, $\times_{lex}$ is the lexicographic product and Γ is well known Mundici's functor establishing a categorical equivalence between the category of MV-algebras and the category of lattice ordered groups with strong unite. 6) says that $\mathbf{c} \neq 0$ and $(C, \mathbf{c})$ is the only subdirectly irreducible algebra; here we have analogy with Boolean algebras.

Hereinafter we denote a CL-algebra as $(A, \mathbf{c})$, where A is an MV-algebra. Denote the class of all CL-algebras by **CL**. We assume that **CL** contains one-element CL-algebra.

Theorem 1. *1) The class* $\boldsymbol{CL}$ *is a quasivariety.*

2) $\boldsymbol{CL}$ *is generated by* $(C, \boldsymbol{c})$*. Moreover,* $\boldsymbol{CL} = \boldsymbol{SP}(C, \boldsymbol{c})$*, where* $\boldsymbol{S}$ *is the operator of taking a subalgebra and* $\boldsymbol{P}$ *is the operator of taking a direct product.*

3) Let $(A, \boldsymbol{c})$ *be a totally ordered* CL*-algebra. There is no element* $x \in A$ *such that* $n\boldsymbol{c} < x < (n+1)\boldsymbol{c}$ *for some* $n \in Z^+$*.*

4) The CL*-algebra* $(C, \boldsymbol{c})$ *is a subalgebra of every* CL*-algebra* $(A, \boldsymbol{c})$*.*

5) Any CL*-algebra* $(A, \boldsymbol{c})$ *is represented as a subdirect product of copies of* CL*-algebras* $(C, \boldsymbol{c})$*.*

3 CL- and CL^+-logics

Now we define a logic CL, the algebraic counterparts of which are CL-algebras. The language of CL consists of the propositional variables, propositional constant c and logical connectives $\to, \neg$. The formulas are defined as usual. The following formulas are axioms of CL:

L1. $\alpha \to (\beta \to \alpha)$,
L2. $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$,
L3. $(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$,
L4. $((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$,
Lp. $2(\alpha^2) \leftrightarrow (2\alpha)^2$,
CL. $c \to \neg c$,
CL1. $2c \odot \neg c \leftrightarrow c$,
CL2. $(c \to \alpha) \vee \neg\alpha$.

Inference rules: MP. $\alpha, \alpha \to \beta \Rightarrow \beta$, R1. $\alpha \vee \neg c \Rightarrow \alpha$,
R2. $\neg(\beta \to \neg\beta) \to \neg((\beta \to \neg\beta) \to (\beta \to \alpha)) \Rightarrow \neg\alpha \to \alpha$.

Semantically, we say that a formula α is 1-true if it is identically true in every CL-algebra, and α is q-true if $\neg\alpha \to \alpha$ is 1-true.

Theorem 2. *1) α is 1-true iff $\vdash_{CL} \alpha$. Hence, α is q-true iff $\vdash_{CL} \neg\alpha \to \alpha$.*
2) $\vdash_{CL} 2(\neg c)^n$ for any $n \in Z^+$.

Let CL^+ be the extension of the logic CL by the inference rule
R4:$(\alpha \vee \neg\alpha) \to \alpha \Rightarrow \alpha$.

Theorem 3. *α is q-true iff $\vdash_{CL^+} \alpha$.*

For every formula α of the logic CL^+ define its translation $tr(\alpha)$ into classical logic Cl as follows: (1) if α is a propositional variable p, then $tr(\alpha) = \alpha$; (2) $tr(c) = p \wedge \neg p$; (3) $tr(\alpha \to \beta) = tr(\alpha) \to tr(\beta)$; (4) $tr(\neg\alpha) = \neg tr(\alpha)$.

Theorem 4. *1) $\vdash_{CL^+} \alpha$ iff $\vdash_{Cl} tr(\alpha)$. 2) If $\vdash_{Cl} \alpha$, then $\vdash_{CL^+} \alpha$.*

References

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