Axiomatising the dual of compact Hausdorff spaces

Luca Reggio
Joint work with Vincenzo Marra

Università degli Studi di Milano,
Dipartimento di Matematica "Federigo Enriques",
via Cesare Saldini 50, Milano, Italy

In 1969 Duskin proved that the dual of the category KHaus of compact Hausdorff spaces and continuous maps is monadic over Set [4, 5.15.3]. This result, from a logical point of view, has two different consequences: on the one hand, it tells us that $\mathsf{KHaus}^{\mathsf{op}}$ is equationally axiomatisable in a (possibly infinitary) algebraic language; on the other hand, that it is axiomatisable in some extension of ordinary first-order logic.

Regarding the algebraic language, it is a classical result in category theory that categories which are monadic over Set coincide, up to equivalence, with varieties of (possibly infinitary) algebras (see [7], or [8, Theorem 5.40 p. 66, Theorem 5.45 p. 68]). The question arises whether $\mathsf{KHaus}^{\mathsf{op}}$ is equivalent to a finitary variety of algebras. The answer is no, and this can be proved by showing that the endofunctor of the induced monad on Set is not finitary (recall that a functor is *finitary* if it preserves directed colimits, and that a category monadic over Set is equivalent to a variety of finitary algebras if, and only if, the associated monad is finitary [1, p. 149]). However Isbell proved [5] that $\mathsf{KHaus}^{\mathsf{op}}$ is equivalent to an $\aleph_1$-variety (that is, a variety in which every function symbol has arity strictly smaller than $\aleph_1$, i.e. at most countable): more precisely, the signature of the latter consists of finitely many finitary operations, along with exactly one operation of countably infinite arity. Indeed, Isbell defined an explicit set of operations and showed that it suffices to generate the algebraic theory of $\mathsf{KHaus}^{\mathsf{op}}$, in the sense of Słomińsky, Lawvere, and Linton [10, 6, 7]; the algebraic theory of $\mathsf{KHaus}^{\mathsf{op}}$ had been described by Negrepontis in [9], by means of Gelfand-Naimark duality between KHaus and the category of commutative unital complex C*-algebras. The problem of axiomatising by equations some $\aleph_1$-variety dually equivalent to KHaus has remained open. Using the theory of MV-algebras as a key tool, along with Isbell's basic insight on the semantic nature of the infinitary operation, we provide a *finite* axiomatisation. Specifically, we introduce a variety of infinitary algebras, that we call δ-algebras, we prove that the category of δ-algebras is a full subcategory of the category of MV-algebras, and we show that it is dually equivalent to KHaus.

With respect to the first-order language, or some extension of it, it is known that $\aleph_1$-varieties are locally $\aleph_1$-presentable categories (this is a particular case of [1, Theorem 3.28]). Recall that, for an infinite regular cardinal λ, an object A of a category is λ-*presentable* if the covariant hom-functor $\hom(A,-)$ preserves λ-directed colimits. A category is said to be λ-*accessible* if it admits a set $\mathcal{A}$ of

λ-presentable objects, such that every object of the category is a λ-directed colimit of objects from $\mathcal{A}$. The category is *locally λ-presentable* if it is λ-accessible and cocomplete. In turn, locally λ-presentable categories are precisely, up to equivalence, the categories of models of certain theories (called *limit theories*) in the infinitary language $L_{\lambda,\lambda}$ [1, Theorem 5.30]. The latter is an extension of first-order logic that allows (finitary function symbols and relation symbols, and) conjunctions and disjunctions of sets of formulæ of cardinality smaller than λ, as well as existential and universal quantification over sets of variables of cardinality smaller than λ. This shows that $\mathsf{KHaus}^{\mathsf{op}}$ can be axiomatised with a (limit) theory in a language that allows countable conjunctions, disjunctions, and quantifications. It is not known whether $\mathsf{KHaus}^{\mathsf{op}}$ is axiomatisable with a first-order theory. Nevertheless, Banaschewski proved [2, p. 1116] that any full subcategory of KHaus, extending the category St of zero-dimensional compact Hausdorff spaces and continuous maps, whose dual is (equivalent to) a class of first-order structures closed under products, has to coincide with St. In particular, $\mathsf{KHaus}^{\mathsf{op}}$ is not axiomatisable with a first-order theory whose category of models is closed under products. In this direction, we prove that any full subcategory of KHaus extending St, whose dual is (equivalent to) an $\aleph_0$-accessible category, has to coincide with St. Since $\aleph_0$-accessible categories are, up to equivalence, precisely the categories of models of geometric theories of presheaf type (see for example [3, Proposition 0.1]), it follows that $\mathsf{KHaus}^{\mathsf{op}}$ is not axiomatisable with a theory of presheaf type. We remark that Banaschewski's result and ours are not comparable: neither of them implies the other.

References

1. Jiří Adámek and Jiří Rosický, *Locally presentable and accessible categories*, London Mathematical Society Lecture Note Series, vol. 189, Cambridge University Press, Cambridge, 1994.
2. Bernhard Banaschewski, *More on compact Hausdorff spaces and finitary duality*, Canad. J. Math. **36** (1984), no. 6, 1113–1118.
3. Tibor Beke, *Theories of presheaf type*, J. Symbolic Logic **69** (2004), no. 3, 923–934.
4. John Duskin, *Variations on Beck's tripleability criterion*, Reports of the Midwest Category Seminar, III, Springer, Berlin, 1969, pp. 74–129.
5. John Isbell, *Generating the algebraic theory of $C(X)$*, Algebra Universalis **15** (1982), no. 2, 153–155.
6. F. William Lawvere, *Functorial semantics of algebraic theories*, Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 869–872.
7. Fred E. J. Linton, *Some aspects of equational categories*, Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965), Springer, New York, 1966, pp. 84–94.
8. Ernest G. Manes, *Algebraic theories*, Springer-Verlag, New York-Heidelberg, 1976, Graduate Texts in Mathematics, No. 26.
9. Joan W. Negrepontis, *Duality in analysis from the point of view of triples*, J. Algebra **19** (1971), 228–253.
10. J. Słomiński, *The theory of abstract algebras with infinitary operations*, Rozprawy Mat. **18** (1959), 67 pp. (1959).