New properties of Sasaki projections

Jeannine J.M. Gabriëls[1], Stephen M. Gagola III[2], and Mirko Navara[3]

[1] Center for Machine Perception, Department of Cybernetics,
Faculty of Electrical Engineering, Czech Technical University in Prague,
Technická 2, 166 27 Praha, Czech Republic
nine.gabriels@gmx.de

[2] School of Mathematics, University of the Witwatersrand,
2050 Johannesburg, South Africa
stephen.gagolaiii@wits.ac.za

[3] Center for Machine Perception, Department of Cybernetics,
Faculty of Electrical Engineering, Czech Technical University in Prague,
Technická 2, 166 27 Praha, Czech Republic
navara@cmp.felk.cvut.cz
http://cmp.felk.cvut.cz/~navara/

With the emergence of quantum mechanics early in this last century, the demand for a mathematical foundation has been raised. In 1936, G. Birkhoff and J. von Neumann published the paper *The logic of quantum mechanics* [2], which got little notice at first. They developed the logic of quantum mechanics by studying the structure of the lattice of projection operators on a Hilbert space $\mathcal{H}$. This lattice constitutes an *orthomodular lattice* (OML for short) which, in general, is not distributive. An OML L is a bounded lattice with a unary operation (*orthocomplementation*) $': L \to L$, $x \mapsto x'$, satisfying:

x' is the lattice-theoretical complement of x	$x \wedge x' = \mathbf{0} \quad x \vee x' = \mathbf{1}$,
$'$ is order-reversing	$x \leqslant y \implies y' \leqslant x'$,
$'$ is an involution	$x'' = x$,
orthomodular law	$x \leq y \implies y = x \vee (x' \wedge y)$.

The *Sasaki projection* was introduced by U. Sasaki [9] and it turns out to be the orthogonal projectors in the Baer *-semigroup of hemimorphisms[4] on the underlying orthomodular lattice (Foulis [5]). It plays an important role and has many interesting properties, some can be found in [1, 3, 4, 8]. In particular, the study [3] has collected deep results which were not superseded for many years. Here we announce further progress and generalization of previous results.

Let L be an OML and $p \in L$. The Sasaki projection φ_p is defined as follows

$$\varphi_p : L \to L$$
$$y \mapsto p \wedge (p' \vee y).$$

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[4] Let L be a OML, $\phi : L \to L$ is a *hemimorphism* if for $x, y \in L$, $\phi(x \vee b) = \phi(x) \vee \phi(y)$ and $\phi(0) = 0$. It follows that ϕ is monotone and submultiplicative, i.e. $\phi(x \wedge y) \leq \phi(x) \wedge \phi(y)$ [5].

The Sasaki projection can also be defined as a binary operation

$$* : L \times L \to L$$
$$(x, y) \mapsto x \wedge (x' \vee y).$$

Here we consider this binary operation and its dual. These operations have been studied e.g. by Beran [1] as a possible substitute for the lattice operations $\wedge$ and $\vee$. These lattice operations are idempotent, commutative and associative, but do not satisfy distributivity. Likewise, the Sasaki operations are idempotent, but do they not satisfy the commutative, associative or distributive laws. They do at least satisfy the identities known from *alternative algebras*:

Proposition 1. *[6]*

$$x * (x * y) = (x * x) * y = x * y, \tag{L}$$
$$(x * y) * y = x * (y * y) = x * y, \tag{R}$$
$$(x * y) * x = x * (y * x) = x * y, \tag{F}$$

along with $x * (x' * y) = (x * x') * y = \mathbf{0}$.

The fundamental observation [7] is that kernels of congruences in an OML L are exactly the subsets $I \subset L$ satisfying

$x * y \in I$ whenever $x \in I$ or $y \in I$.

The meet, $x \wedge y$, does not possess this property.

We consider equations like

$$a * x = b \quad \text{or} \quad x * a = b,$$

where a, b are fixed elements of an OML and x is the unknown. Unlike for the typical lattice operations, the sets of solutions for the Sasaki operations may be rather non-trivial.

Furthermore, we prove equations inspired by the identities fulfilled in octonion algebras. Although such identities hold only under additional assumptions, we have the following results:

Proposition 2. *[6, Theorem 5.3]*

$$(x * y * x) * z = (x * y) * (x * z), \tag{1}$$
$$(z * (x * y)) * x = z * (x * y * x), \tag{2}$$
$$((x * y) * z) * x = (x * y) * (z * x), \tag{3}$$

where a lack of parentheses $x * y * z$ means that the order of multiplication does not matter, i.e. the expression fulfills the associativity equation.

As observed before, Sasaki operations form a promising alternative to lattice operations (join and meet) in the study of orthomodular lattices. We bring arguments that Sasaki operations satisfy more equations than other operations

(except for the lattice operations) and therefore they are the best candidates for developing alternative algebraic tools for computations in orthomodular lattices. We conclude that the potential of using Sasaki operations in the algebraic foundations of orthomodular lattices is still not sufficiently exhausted.

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