Bisimulation and path logic for sheaves

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In the seminal paper [2] Joyal, Winskel and Nielsen offered a general categorical notion of bisimulation encompassing the notions of behavioural equivalence occurring in different models of concurrency. This notion relied on a representation of models of concurrency in terms of presheaves, and it consisted essentially in a span of open maps, namely morphisms with a special path-lifting property. In the same work they proposed a logic that is characteristic for such bisimulations, a modal logic baptized with the name *path logic*. The idea behind this logic is to view arrows in the base category as labels for modalities, and to view a presheaf as a labelled transition system as in [6].

This talk focuses on *sheaves* of sets over topological spaces, as a special case of presheaves.[3] It is well known that the original reason for defining and studying sheaves came from algebraic topology [3]. On the other hand, there exists a wealth of connections between topological spaces and modal logics (see [5] for an extensive survey). Therefore it seems natural to investigate the expressive power of path logic when restricted to sheaves, in particular to sheaves with a topological significance, and to relate the logic with an apt notion of sheaf bisimulation.

As one goes about investigating the expressive power of path logic over sheaves, the first step is a thorough understanding of the expressive power of path logic over presheaves. It was shown in [2] that path logic characterizes *path bisimulation*, which is a concrete notion of bisimulation that is equivalent to the existence of a *span of open maps*. We add to this result the observation that, for the *finitary* version of the path logic, a van Benthem-style characterization theorem can be obtained with respect to a multi-sorted first-order language. This result can be exploited to obtain further interesting observations. For example, it turns out that in some instances the correspondence between the many sorted **FOL** given by such category and the path logic is decidable. This is somewhat surprising since in regular Correspondence Theory it is known that bisimulation invariance for **FOL**-formulas is not decidable.

In the context of sheaves, it turns out that the basic path logic is too weak. In particular, the property of being a sheaf itself cannot be captured by any axiom of modal path logic. To remedy this issue, we extend the modal language with a minimal amount of hybrid machinery, i.e. hybrid constants. We call the resulting language *hybrid path logic*. In this logic, we can give a simple set of axioms that capture the notion of a sheaf over a fixed base space:

[3] Recall that a sheaf is a presheaf satisfying the additional conditions known as 'locality' and 'gluing' [3].

Theorem 1. *The class of sheaves over a given base space $\mathbb{X}$ is frame definable in the hybrid path logic corresponding to $\mathbb{X}$.*

As our prime example of a sheaf over a topological space, we consider the sheaf of sections of a covering space $\pi : \mathbb{C} \to \mathbb{X}$ [3]. This sheaf gives rise to a transition system whose points are the sections themselves. We can then describe some non-trivial properties of the covering space $\mathbb{C}$ in the path logic. For example, we can capture the existence of a global section, a notion that turned out to be central in the sheaf-theoretic analysis of non-locality and contextuality in quantum and classical phenomena [1].

Finally, we address the issue of bisimulations on sheaves. The characterization of the notion of bisimulation in the case of presheaves hangs on the relation between path bisimulation, a notion of bisimulation closer to the one on transition system, and its categorical counterpart in terms of spans of open maps.

Motivated by this, we investigate whether path bisimulations *on sheaves* can be represented by some purely categorical concept. Here, matters appear to be more subtle than in the general case for presheaves, since we are after a categorical construct that 'lives' within the category of sheaves. A span of open maps in the category of sheaves corresponds to a path bisimulation, but we do not know whether the converse holds. If we strengthen the definition of path bisimulation with a condition called "Locality", because of its likeness to the corresponding condition on sheaves, then the existence of a path bisimulation does entail the existence of a span of open maps. For this case we do not know whether the other direction holds, i.e. whether any bisimulation satisfying Locality gives rise to a span of open maps.

Taking some inspiration from Universal Coalgebra [4], we consider *co-spans* of open maps instead of spans.[4] To capture the additional structure on sheaves we further enrich the notion of path bisimulation with a condition called "Glueing", again in analogy with the homonym property of sheaves. Here, we do get an exact correspondence:

Theorem 2. *The existence of a path bisimulation satisfying Locality and Glueing is equivalent to the existence of a co-span of open maps.*

The question whether a span of open maps determines the existence of such bisimulations is still open, although this is indeed the case in some special circumstances. The converse, on the other hand, is secured by the closure of open maps under pullbacks.

References

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[4] This corresponds to what is known as *behavioural equivalence* in the coalgebra community.

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