Full Lambek Calculus with contraction is undecidable*

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Among propositional substructural logics, these obtained from Gentzen's sequent calculus for intuitionistic logic (**LJ**) by removing a subset of the rules contraction (c), exchange (e), left weakening (i), and right weakening (o) play a prominent role, e.g. in [3] such logics are called basic substructural logics. If all above mentioned rules are removed from **LJ** then the full Lambek calculus is obtained.

The decidability of such logics, i.e. their sets of theorems, usually follows from the fact that they have a cut-free sequent system. Such an argument, used in [8], however, fails if the rule of contraction is involved since the proof-search tree is then infinite. Nevertheless, already Gentzen proved [4, 5] that **LJ** is decidable and the same was shown [7] for **FL** with the rules of exchange and contraction ($\mathbf{FL_{ec}}$) using an idea by Kripke [9]. It remained open whether same holds for **FL** with contraction ($\mathbf{FL_c}$) and **FL** with contraction and right weakening ($\mathbf{FL_{co}}$). We show that these logics are, on the contrary, undecidable by showing that their common positive fragment ($\mathbf{FL_c^+}$) is already undecidable.

In fact, we show that the equational theory of square-increasing residuated lattices ($\mathcal{RL}_c$), which are sound and complete algebraic semantics for $\mathbf{FL_c^+}$, is undecidable. However, the algebraic notions were used only for convenience, the whole construction can be shown using, e.g. proof-theoretical notions, because the main ideas remain the very same.

Theorem 1. *The equational theory of $\mathcal{RL}_c$ is undecidable. Consequently, the sets of formulae provable in $\mathbf{FL_c^+}$, $\mathbf{FL_c}$, and $\mathbf{FL_{co}}$ are undecidable.*

Note that this is not very common among known substructural logics. The undecidability of the positive fragment of the involutive distrubutive $\mathbf{FL_{ec}}$ is proved in [10] and the same for the equational theory of modular lattices is shown in [2].

In what follows, we give the main ideas of the proof. It was proved in [6] that the deducibility problem for $\mathbf{FL_c^+}$ is undecidable using a string rewriting system (SRS) which simulates Minsky machines by square-free words, i.e. the rule of contraction cannot affect them.

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This SRS is then equivalently expressed in terms of atomic conditional SRSs which differ from SRSs in two aspects. First, only rules with atomic right side are allowed, i.e. $x \rightsquigarrow a$ where $x \in \Sigma^*$ and $a \in \Sigma$. Second, the usage of every rule is restricted by a specific context in which it is applicable.

Finally, an encoding of atomic conditional SRSs in $\mathcal{RL}_c$ is shown. Roughly speaking the conditionality in rules is expressed by join and an auxiliary rewriting system (inspired by [1]), the rewriting symbol $\rightsquigarrow$ is encoded by an implication and a set of rules by a meet of encoded rules. Although the constant 1 plays also an important role in this encoding, it can be shown that it is not necessary. Therefore even the fragments of $\mathcal{RL}_c$ and $\mathbf{FL}_\mathbf{c}^+$ containing only join, meet, and an implication are undecidable.

We conclude with some notes. The whole construction can be easily modified for logics having a weaker form of contraction $x^k \leq x^l$, $1 \leq k < l$. More interestingly, as the construction, in fact, provides a chain of explicit reductions, it is possible to obtain a form of "algorithmic" deduction theorem.

Theorem 2. *Let $T \cup \{\varphi\}$ be a finite set of formulae. There is an explicit algorithm that produces a formula ψ (given an input φ and T) such that ψ is provable in $\mathbf{FL}_\mathbf{c}^+$ iff φ is provable in $\mathbf{FL}_\mathbf{c}$ from T.*

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