Endomorphism monoids of ω-categorical structures

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Abstract. Two omega-categorical structures are first order bi-interpretable iff their automorphism groups are isomorphic as topological groups, endowed with the topology of pointwise convergence. Does the theorem still hold, when we request the automorphism groups to be only algebraically isomorphic? In 1990 Evans and Hewitt constructed an example that negates this question. In my talk I would like to present their example and show that also the endomorphism monoids, respectively the polymorphism clones of those two omega-categorical structures are isomorphic, but not topologically isomorphic. This answers the same question on some more refined versions of bi-interpretability.

Keywords: omega-categorical structures, interpretability, reconstruction.

A countable structure Γ is called *ω-categorical* if its theory has, up to isomorphism, exactly one countable model. There is a close connection between ω-categorical structures and their automorphism groups: By the theorem of Ryll-Nardzewski a structure is ω-categorical if and only if its automorphism group is an *oligomorphic permutation group*, meaning that the action of $\mathrm{Aut}(\Gamma)$ on every finite power of Γ has only finitely many orbits.

We can also regard the automorphism groups as topological groups, bearing the topology of pointwise convergence. Ahlbrandt and Ziegler showed that two ω-categorical structures Γ_1, Γ_2 are first order bi-interpretable, if and only if their automorphism groups are isomorphic as topological groups ([3]).

One might ask, if the theorem of Ahlbrandt and Ziegler still holds when we request the automorphism groups to be only algebraically isomorphic. So, if there is an isomorphism $\mathrm{Aut}(\Gamma_1) \to \mathrm{Aut}(\Gamma_2)$, is there also a topological isomorphism? This is a valid question, because for a lot of well-known ω-categorical structures, for example the countable set, the rational order $(\mathbb{Q}, <)$ and every countable ω-categorical abelian group, the answer is yes. In fact, all these structures satisfy an even stronger property, the *small index property*: The open subgroups of their automorphism group are exactly the subgroups of at most countable index. If Γ_1 has the small index property it is easy to see, that *every* isomorphism $\mathrm{Aut}(\Gamma_1)$

$\to \mathrm{Aut}(\Gamma_2)$ is a homeomorphism.

In 1990 Evans and Hewitt published an example of two ω-categorical structures, whose automorphism groups are isomorphic, but not topologically isomorphic ([1]). A central part of their proof is that every separable profinite group is topologically isomorphic to the quotient group of two oligomorphic groups. Using this result and the existence of two separable profinite groups, that are isomorphic but not topologically isomorphic, they were able to construct the counterexample.

In the last year some variants of the theorem of Ahlbrandt and Ziegler have been shown. We call two structures *existential positive bi-interpretable* if they are bi-interpretable by the means of existential positive formulas, i.e. formulas without universal quantifiers, equalities and negation symbols. A result of Bodirsky and Junker ([4]) states that (under some additional technical conditions) two ω-categorical structures are existential positive bi-interpretable if and only if their endomorphism monoid are topologically isomorphic.

We call two structures *primitively positive bi-interpretable* if they are bi-interpretable by the means of primitive positive formulas, i.e. only by using conjunctions and existential quantifiers. For a structure Γ we define the *polymorphism clone* $\mathrm{Pol}(\Gamma)$ to consist of all finitary operations $f : \Gamma^n \to \Gamma$, which preserve all the relations on Γ. Now two ω-categorical structures are primitively positive bi-interpretable, if and only if their *polymorphism clones* are topologically isomorphic ([5]).

Again the question arises, if those theorems still hold, when we forget about the topology. With some additional thought on the example of Evans and Hewitt, we are able to show that also the topology of endomorphism monoids and polymorphism clones of ω-categorical structures can't be reconstructed from their bare algebraic structure.

References

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