Classification of absorbent-continuous, densely ordered, complete, group-like FL_e-chains

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Abstract. The main result of the present paper is the classification of absorbent-continuous, order-dense group-like FL_e-chains: Every such algebra can be described as the twin rotation of a certain BL-algebra and its de Morgan dual. This theorem not only largely generalizes the corresponding main theorem of [17, 15] but also the complexity and length of its proof is significantly reduced.

Residuated *semigroups* have been introduced in the 30s of the last century by Ward and Dilworth [23] to investigate ideal theory of commutative rings with unit. Examples of residuated lattices include Boolean algebras, Heyting algebras, complemented semigroups [6], bricks [4], residuation groupoids [7], semiclans [5], Bezout monoids [3], MV-algebras [8], BL-algebras [12], and lattice-ordered groups. Residuated lattices play a crucial role in the algebraization of substructural logics [11]. As for the *classification* of residuated lattices, as one naturally expects, this is possible only by imposing additional postulates. A precursor is due to Hölder, who proved in [13] that every *cancellative*, *Archimedean*, naturally and totally ordered semigroup can be embedded into the additive semigroup of the real numbers. Aczél used tools of analysis to investigate continuous semigroup operations over intervals of real numbers and also found in [1, page 256] the cancellative property to be sufficient and necessary for the existence of an order-isomorphism to a subsemigroup of the additive semigroup of the real numbers [1, page 268]. Clifford showed in [9] that every *Archimedean*, naturally and totally ordered semigroup in which the *cancellation law does not hold* can be embedded into either the real numbers in the interval $[0, 1]$ with the usual ordering and $ab = \max(a + b, 1)$ or the real numbers in the interval $[0, 1]$ and the symbol ∞ with the usual ordering and $ab = a + b$ if $a + b \leq 1$ and $ab = \infty$ if $a + b > 1$. For a summary of the Hölder and Clifford theorems, see [10, Theorem 2 in Section 2 of Chapter XI]. Clifford introduced also the ordinal sum construction for a family of totally ordered semigroups in [9] and proved that every naturally totally ordered, commutative semigroup is uniquely expressible as the ordinal sum of a totally ordered set of ordinally irreducible semigroups of this kind. Mostert and Shields gave a complete description of topological semigroups over compact manifolds with connected, regular boundary in [22] by using a subclass of compact connected Lie groups and via classifying semigroups on arcs such that one endpoint is identity for the semigroup, and the other is zero. They classified such semigroups as ordinal sums of three basic multiplications which an arc may possess. The word 'topological' refers to the continuity of the semigroup operation with respect to the topology. In the next related classification result, the property of topological connectedness of the underlying chain was dropped whereas the continuity condition was somewhat strengthened: Under the assumption of divisibility[1], residuated chains were classified as ordinal sums of linearly ordered Wajsberg hoops in [2]. Postulating the divisibility condition proved to be sufficient for the classification of commutative, integral, prelinear residuated monoids over arbitrary lattices, see [20], where the authors introduced the notion of poset sum of hoops, a common generalization of ordinal sum and of direct product. They proved that certain GBL-algebras[2] embed into the poset sum of a family of MV-chains and that the embedding is an isomorphism in the finite case. Next, SIU-chains were classified in [18]; here the authors assume the existence of a dual isomorphism between the positive and negative cones of the algebra. Over densely ordered, complete chains it is equivalent to postulating divisibility only for the negative cone of the algebra. Absorbent-continuous, group-like FL_e-algebras over weakly real[3] chains have been classified in [17, 15]. Absorbent continuity can be seen as an extremely relaxed version of the divisibility condition. Finally, absorbent-continuous, densely ordered and complete, group-like FL_e-chains have been classified in [14] as follows: We call an FL_e-monoid group-like if $t = f$. We call a group-like FL_e-monoid and also its monoidal operation $\circledast$ *absorbent-continuous* if for all x from its negative cone, the absorbing set $\{z : z \circledast x = x\}$ of x has its least element.

Theorem 1. *$\mathcal{U}$ is an absorbent-continuous, densely-ordered, complete, group-like FL_e-chain with involution $'$ if and only if $\mathcal{U}$ is the twin-rotation ([19]) of a BL-algebra and its de Morgan dual with respect to $'$, where the BL-algebra has components ([2]) which are either cancellative[4] or MV-algebras over two elements, and the BL-algebra cannot have two consecutive cancellative components.*

Theorem 1 doesn't hold if the chain is not order dense or not complete. Also, the dropping of either absorbent-continuity or the group-like condition yields algebras other than that of Theorem 1. The result seems to be very surprising since quite weak conditions characterize a quite specific class of algebras. We shall present the main steps of its proof, and meanwhile emphasize the role and the strength of the geometry of associativity [16]. Generalization of the result by dropping absorbent-continuity would largely contribute to the solution of the

[1] Divisibility is the dual notion of being naturally ordered. For residuated integral monoids, if the underlying chain is order dense and complete then divisibility is equivalent to the continuity of the semigroup operation in the order topology.

[2] BL-algebras are particular commutative GBL-algebras.

[3] There is a flaw in the proof of the main theorem in [17], namely, absorbent continuity is not preserved by MacNeille completion, see [15]. Therefore, the theorem holds for weakly real chains and not for subreal chains.

[4] That is, those components are negative cones of totally ordered Abelian groups.

open problem related to the standard completeness of Involutive Uninorm Logic [21].

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