On weak constant domain principle in the Kripke sheaf semantics

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Abstract. We consider superintuitionistic predicate logics understood in the usual way, as sets of predicate formulas (without equality and function symbols) containing all axioms of Heyting predicate logic **Q-H** and closed under modus ponens, generalization, and substitution of arbitrary formulas for atomic ones.

1 We consider the semantics of predicate Kripke frames with equality (called *e-frames*, for short), which is equivalent to the semantics of Kripke sheaves (see e.g. [1] or [2]). Namely, an *e-frame* is a triple $M = (W, U, I)$ formed by a poset W with the least element 0_W, a domain map U defined on W such that $\varnothing \neq U(u) \subseteq U(v)$ for $u \leq v$, and a family I of equivalence relations I_u on $U(u)$ for $u \in W$ such that $I_u \subseteq I_v$ for $u \leq v$. A usual *(predicate) Kripke frame* is an e-frame with equalities I_u (i.e., $aI_ub \Leftrightarrow a = b$ for $u \in W$, $a, b \in U(u)$).

A *valuation* $u \vDash A$ (for $u \in W$ and formulas A with parameters replaced by elements of $U(u)$) satisfies the monotonicity: $u \leq v,\ u \vDash A \ \Rightarrow\ v \vDash A$, the usual inductive clauses for connectives and quantifiers, e.g.

$$u \vDash (B \to C) \ \Leftrightarrow\ \forall v \geq u\, [(v \vDash B) \Rightarrow (v \vDash C)],$$

$$u \vDash \forall x\, B(x) \ \Leftrightarrow\ \forall v \geq u\, \forall c \in U(v)\, [v \vDash B(c)],$$

etc., and preserves I_u (on every $U(u), u \in W$), i.e.,

$$\bigwedge_i (a_i\, I_u\, b_i) \ \Rightarrow\ (\, u \vDash A(a_1, \ldots, a_n) \ \Leftrightarrow\ u \vDash A(b_1, \ldots, b_n)\,).$$

A formula $A(\mathbf{x})$ (where $\mathbf{x} = (x_1, \ldots, x_n)$) is *valid* in M if it is true under any valuation in M, i.e., if $u \vDash A(\mathbf{a})$ for any $u \in W$ and $\mathbf{a} \in (D_u)^n$. The *predicate logic* $\mathbf{L}(M)$ *of an (e-)frame* M is the set of all formulas valid in M.

2 We consider the constant domain principle

$$D = \forall x(P(x) \vee Q) \to \forall x P(x) \vee Q$$

(where P and Q are unary and 0-ary symbols, respectively), and its weak ('negative') version

$$D^- = \forall x\, (\neg P(x) \vee Q) \to \forall x \neg P(x) \vee Q\,.$$

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The formula D states (in an e-frame) that $\forall a\in U(u)\,\exists b\in U(0_W)\,[\,aI_u b\,]$, and similarly, D^- states that $\forall a\in U(u)\,\exists b\in U(0_W)\,[\,\exists v\geq u\,(\,aI_v b\,)\,]$.
Let D^--*frames* be e-frames satisfying the latter condition, i.e., validating D^-.

Clearly, $D\vdash D^-$ (we write $A\vdash B$ for $[\mathbf{Q\text{-}H}+A]\vdash B$). Also:

D is valid in M iff D^- is valid in M iff $U(u)=U(0_W)$ for every $u\in W$ for a usual Kripke frame M. Hence the Kripke-completion of $[\mathbf{Q\text{-}H}+D^-]$ is $[\mathbf{Q\text{-}H}+D]$. Now we describe the Kripke sheaf completion of $[\mathbf{Q\text{-}H}+D^-]$.

3 We consider the following formulas (for $n>0$, $m\geq 0$):

$$\begin{aligned}
D^-_{n,m} = {} & \forall z(Q_0\vee P_0(z))\&\forall x R(x,x) \;\to \\
& \to\; Q_0\vee\forall \mathbf{x}_0\,[\,\forall z(P_0(z)\to Q_1(\mathbf{x}_0)\vee P_1(\mathbf{x}_0,z))\to \\
& \to Q_1(\mathbf{x}_0)\vee\forall \mathbf{x}_1\,[\,\forall z(P_1(\mathbf{x}_0,z)\to Q_2(\mathbf{x}_0,\mathbf{x}_1)\vee P_2(\mathbf{x}_0,\mathbf{x}_1,z))\to \\
& \to \ldots \\
& \to Q_{n-2}(\mathbf{x}_0,\ldots,\mathbf{x}_{n-3})\vee\forall \mathbf{x}_{n-2}\,[\,\forall z(P_{n-2}(\mathbf{x}_0,\ldots,\mathbf{x}_{n-3},z)\to \\
& \qquad \to\; Q_{n-1}(\mathbf{x}_0,\ldots,\mathbf{x}_{n-2})\vee P_{n-1}(\mathbf{x}_0,\ldots,\mathbf{x}_{n-2},z))\to \\
& \to Q_{n-1}(\mathbf{x}_0,\ldots,\mathbf{x}_{n-2})\vee\forall \mathbf{x}_{n-1},y\,[\,\forall z(P_{n-1}(\mathbf{x}_0,\ldots,\mathbf{x}_{n-2},z)\to \\
& \qquad \to\; Q_n(\mathbf{x}_0,\ldots,\mathbf{x}_{n-1},y)\vee\neg R(y,z))\to Q_n(\mathbf{x}_0,\ldots,\mathbf{x}_{n-1},y)]]\ldots]].
\end{aligned}$$

Here P_i are $(1+m\cdot i)$-ary predicate symbols (for $0\leq i<n$), Q_i are $(m\cdot i)$-ary symbols (for $0\leq i<n$), Q_n is a $(1+m\cdot n)$-ary symbol, R is a binary symbol; also $\mathbf{x}_i=(x_{i,1},\ldots,x_{i,m})$ (for $0\leq i<n$) are disjoint lists of different variables, and x,y,z are different variables non-occurring in $\mathbf{x}_0,\ldots,\mathbf{x}_{n-1}$.

It can be easily shown that $D^-_{n,m}\vdash D^-_{n',m'}$ for $n\geq n', m\geq m'$ and $D^-_{1,0}\vdash D^-$. Moreover,

$$(\mathbf{Q\text{-}H}+D^-)\;\subset\;(\mathbf{Q\text{-}H}+\{D^-_{n,m}:n>0,m\geq 0\})\;=\;(\mathbf{Q\text{-}H}+\{D^-_{n,n}:n>0\}).$$

Also one can show that the formulas $D^-_{n,m}$ are valid in all D^--frames. Thus:

$D^-_{n,m}$ is valid in an e-frame M iff D^- is valid in an e-frame M, i.e., iff M is a D^--frame (for any n,m).

Theorem 1. . *The logic* $(\mathbf{Q\text{-}H}+\{D^-_{n,m}:n>0,m\geq 0\})$ *is complete w.r.t.* D^-*-frames.*

Hence this logic is the Kripke sheaf completion of $(\mathbf{Q\text{-}H}+D^-)$. We believe that this completion is not finitely axiomatizable.

Some related completeness results for extensions with Kuroda's formula $K=\neg\neg\forall x\,(P(x)\vee\neg P(x))$ and with predicate axioms of finite heights P^+_m will be mentioned in the talk (here $P^+_0=\bot$ and $P^+_{n+1}=\forall x\,[\,R_n(x)\vee(R_n(x)\to P^+_n)\,]$ for $n\geq 0$; R_n being different unary predicate symbols).

References

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2. Gabbay, D., V. Shehtman, and D. Skvortsov, *Quantification in nonclassical logic*, Vol. 1, Sections 2.6, 3.6, Studies in Logic and the Foundations of Mathematics 153: Elsevier, 2009.