On weak constant domain principle in the Kripke sheaf semantics

Dmitrij Skvortsov

All-Russian Institute of Scientific and Technical Information, VINITI,
skvortsovd@yandex.ru

Abstract. We consider superintuitionistic predicate logics understood in the usual way, as sets of predicate formulas (without equality and function symbols) containing all axioms of Heyting predicate logic $\mathbf{Q-H}$ and closed under modus ponens, generalization, and substitution of arbitrary formulas for atomic ones.

1 We consider the semantics of predicate Kripke frames with equality (called $e$-frames, for short), which is equivalent to the semantics of Kripke sheaves (see e.g. [1] or [2]). Namely, an $e$-frame is a triple $M = (W, U, I)$ formed by a poset $W$ with the least element $0_W$, a domain map $U$ defined on $W$ such that $\emptyset \neq U(u) \subseteq U(v)$ for $u \leq v$, and a family $I$ of equivalence relations $I_u$ on $U(u)$ for $u \in W$ such that $I_u \subseteq I_v$ for $u \leq v$. A usual (predicate) Kripke frame is an $e$-frame with equalities $I_u$ (i.e., $aI_u b \iff a = b$ for $u \in W$, $a, b \in U(u)$).

A valuation $u \models A$ (for $u \in W$ and formulas $A$ with parameters replaced by elements of $U(u)$) satisfies the monotonicity: $u \leq v$, $u \models A \Rightarrow v \models A$, the usual inductive clauses for connectives and quantifiers, e.g.

\[ u \models (B \to C) \iff \forall v \geq u [(v \models B) \Rightarrow (v \models C)], \]
\[ u \models \forall x B(x) \iff \forall v \geq u \forall c \in U(v) [v \models B(c)], \]

e tc., and preserves $I_u$ (on every $U(u), u \in W$), i.e.,
\[ \bigwedge_i (a_i I_u b_i) \Rightarrow (u \models A(a_1, \ldots, a_n) \iff u \models A(b_1, \ldots, b_n)). \]

A formula $A(x)$ (where $x = (x_1, \ldots, x_n)$) is valid in $M$ if it is true under any valuation in $M$, i.e., if $u \models A(a)$ for any $u \in W$ and $a \in (D_u)^n$. The predicate logic $\mathbf{L}(M)$ of an $(e)$-frame $M$ is the set of all formulas valid in $M$.

2 We consider the constant domain principle
\[ D = \forall x (P(x) \lor Q) \to \forall x P(x) \lor Q \]
(where $P$ and $Q$ are unary and 0-ary symbols, respectively), and its weak (‘negative’) version
\[ D^- = \forall x (\neg P(x) \lor Q) \to \forall x \neg P(x) \lor Q. \]

* This work is supported by the RFBR-CNRS-grant # 14-01-93105.
The formula \( D \) states (in an e-frame) that \( \forall a \in U(u) \exists b \in U(0w) [aI_0b] \), and similarly, \( D^- \) states that \( \forall a \in U(u) \exists b \in U(0w) [\exists v \geq u (aI_vb)] \).

Let \( D^- \)-frames be e-frames satisfying the latter condition, i.e., validating \( D^- \).

Clearly, \( D \vdash D^- \) (we write \( A \vdash B \) for \( \exists \in A \vdash B \)). Also:

\( D \) is valid in \( M \) iff \( D^- \) is valid in \( M \) iff \( U(u) = U(0w) \) for every \( u \in W \) for a usual Kripke frame \( M \). Hence the Kripke-completion of \( [\mathbf{Q} \mathbf{H} + D^-] \) is \( [\mathbf{Q} \mathbf{H} + D] \). Now we describe the Kripke sheaf completion of \( [\mathbf{Q} \mathbf{H} + D^-] \).

We consider the following formulas (for \( n > 0, m \geq 0 \)):

\[
D^-_{n,m} = \forall z(Q_0 \lor P_0(z)) \land \forall x R(x, x) \rightarrow
\rightarrow Q_0 \lor \forall x_0 [\forall z(P_0(z) \rightarrow Q_1(x_0) \lor P_1(x_0, z)) \rightarrow
\rightarrow Q_1(x_0) \lor \forall x_1 [\forall z(P_1(x_0, z) \rightarrow Q_2(x_0, x_1) \lor P_2(x_0, x_1, z)) \rightarrow
\rightarrow \ldots
\rightarrow Q_{n-2}(x_0, \ldots, x_{n-3}) \lor \forall x_{n-2} [\forall z(P_{n-2}(x_0, \ldots, x_{n-3}, z) \rightarrow
\rightarrow Q_{n-1}(x_0, \ldots, x_{n-2}) \lor P_{n-1}(x_0, \ldots, x_{n-2}, z)) \rightarrow
\rightarrow Q_{n-1}(x_0, \ldots, x_{n-2}) \lor \forall x_{n-1}, y [\forall z(P_{n-1}(x_0, \ldots, x_{n-2}, z) \rightarrow
\rightarrow Q_n(x_0, \ldots, x_{n-1}, y) \lor \neg R(y, z)) \rightarrow Q_n(x_0, \ldots, x_{n-1}, y)] \ldots] .
\]

Here \( P_i \) are \((1+m \cdot i)-ary\) predicate symbols (for \( 0 \leq i < n \))\), \( Q_i \) are \((m \cdot i)-ary\) symbols (for \( 0 \leq i < n \))\), \( R \) is a \( 2 \)-ary binary symbol; also \( x_i = (x_{i,1}, \ldots, x_{i,m}) \) (for \( 0 \leq i < n \)) are disjoint lists of different variables, and \( x, y, z \) are different variables non-occurring in \( x_0, \ldots, x_{n-1} \).

It can be easily shown that \( D^-_{n,m} \vdash D^-_{n',m'} \) for \( n \geq n', m \geq m' \) and \( D^-_{1,0} \vdash D^- \). Moreover,

\[
[\mathbf{Q} \mathbf{H} + D^-] \subset [\mathbf{Q} \mathbf{H} + \{D^-_{n,m} : n > 0, m \geq 0 \}) = [\mathbf{Q} \mathbf{H} + \{D^-_{n,n} : n > 0 \}].
\]

Also one can show that the formulas \( D^-_{n,m} \) are valid in all \( D^- \)-frames. Thus:

\( D^-_{n,m} \) is valid in an e-frame \( M \) iff \( D^- \) is valid in an e-frame \( M \), i.e., iff \( M \) is a \( D^- \)-frame (for any \( n, m \)).

**Theorem 1.** The logic \( [\mathbf{Q} \mathbf{H} + \{D^-_{n,m} : n > 0, m \geq 0 \}) \) is complete w.r.t. \( D^- \)-frames.

Hence this logic is the Kripke sheaf completion of \( [\mathbf{Q} \mathbf{H} + D^-] \). We believe that this completion is not finitely axiomatizable.

Some related completeness results for extensions with Kuroda’s formula \( K = \neg \forall x(P(x) \lor \neg P(x)) \) and with predicate axioms of finite heights \( P^n \) will be mentioned in the talk (here \( P_0^+ = \perp \) and \( P_{n+1}^+ = \forall x [R_n(x) \lor (R_n(x) \rightarrow P^n_+)] \)

for \( n \geq 0 \); \( R_n \) being different unary predicate symbols).

**References**
