Projective Unification in Intermediate and Modal Predicate Logics

Wojciech Dzik[1] and Piotr Wojtylak[2]

[1] Institute of Mathematics, Silesian University, Katowice, Poland
wojciech.dzik@us.edu.pl

[2] Institute of Mathematics and Computer Science, University of Opole, Poland
piotr.wojtylak@math.uni.opole.pl

Abstract. Projective unifiers were introduced by S. Ghilardi and successfuly applied in propositional logic: intuitionistic and modal, see [6], [7], [8], Still new applications are being studied, see e.g. [1], [9]. Our aim is to lift projective unifiers to the first-order (or predicate) level. However extending results on projective unification of some intermediate and modal propositional logics to their predicate counterparts is not immediate and it requires a proper definition of substitution and additional axioms such as the principle of Independence of Premise, *IP*.

Let a first-order modal language without function symbols be given (for details see [2]). If ε is a substitution for predicate variables, it is usual (see [11]) to assume that $vf(\varepsilon(A)) \subseteq vf(A)$ for each formula A, where $vf(\varepsilon(B))$ denotes the set of free variables occurring in B. For our approach, however, such approach would be too restrictive. We need a more general concept where $\varepsilon(P(a_1, \ldots, a_n))$ may contain – in addition to $a_1, \cdots, a_n$ – other free variables. These additional variables are regarded as parameters of the substitution.

If L is an intermediate or modal propositional logic, then Q-L denotes the corresponding predicate logic. Any predicate logic, in addition to many specific conditions (see [2]), must be also closed under the (above mentioned extended concept of) substitution for predicate variables.

Similarly as in propositional logic, a *unifier* for a formula A in a predicate logic L is a substitution ε (for predicate variables) such that $\varepsilon(A)$ is derivable in L, i.e. $\vdash_L \varepsilon(A)$. A formula A is said to be unifiable in L if it has a unifier. A unifier ε for A in L is *projective* if $A \vdash_L \varepsilon(P_i(a_1, \ldots, a_n)) \leftrightarrow P_i(a_1, \ldots, a_n)$ for each predicate variable P_i. Clearly, if ε is projective for A in L, then $A \vdash_L \varepsilon(B) \leftrightarrow B$ for each formula B. We say that a logic L enjoys *projective unification* if each unifiable formula has a projective unifier in L.

An intermediate propositional logic is known to have projective unification (see [12]) iff it extends LC. We manage to extend this result to predicate logics even that Q-LC does not enjoy projective unification. The following principle of *Independence of Premise*, *IP* in short, is known in constructive mathematics and proof theory:

$$(IP) \qquad (A \Rightarrow \exists_x B(x)) \Rightarrow \exists_x (A \Rightarrow B(x)),$$

where x is not free in A. We have

Theorem 1. *An intermediate predicate logic L enjoys projective unification iff $L \supseteq IP.Q\text{-}LC$.*

$IP.Q\text{-}LC$ denotes an extension of $Q\text{-}LC$ with IP, according to the notation of extensions used in [2]. To prove that any L extending $IP.Q\text{-}LC$ enjoys projective unification it is sufficient to modify the propositional unifier as given by [10]. Thus, for each unifiable first-order formula A one receives in a straightforward and uniform way (via the ground uniform method) its projective unifier ε which is a substitution for predicate variables satisfying the condition $vf(\varepsilon(B)) \subseteq vf(B)$ for each formula B .

In case of modal logics the situation is more complicated. Ground unifier method suffices only to show, see [4], that

Theorem 2. *Any modal predicate logic over $Q\text{-}S5$ enjoys projective unification.*

As it is known, see [5], a propositional modal logic enjoys projective unification iff it extends $S4.3$. However, even at the propositional level, projective unifiers cannot be received in a uniform way (using any form of ground unifier method). Nor one should expect that, in predicate logic, unifiers would satisfy the condition $vf(\varepsilon(B)) \subseteq vf(B)$. Clearly, the modal version of (IP), that is

$$(\Box IP) \qquad \Box(A \rightarrow \exists_x \Box B(x)) \rightarrow \exists_x \Box(A \rightarrow B(x))$$

is required for projective unification. Though we were only able to prove

Theorem 3. *Any modal predicate logic $L_=$ with equality enjoys projective unification iff $L_=$ extends $\Box IP.Q\text{-}S4.3_=$.*

References

1. Baader, F.,Ghilardi, S., *Unification in Modal and Description Logics*, **Logic Journal of the IGPL**, 19(5)(2011), 705-730.
2. Braüner, T., Ghilardi, S. *First-order Modal Logic*, Chapter 9 in Blackburn, P., **Handbook of Modal Logic** 2007 Elsevier, 549–620.
3. Dzik, W. *On Structural completeness of some nonclassical predicate calculi*, **Reports on Mathematical Logic**, 5 (1995), 19–26
4. Dzik, W. *Chains of Structurally Complete Predicate Logics with the Application of Prucnal'S Substitution* **Reports on Mathematical Logic**, 38 (2004), 37 –48.
5. Dzik, W., Wojtylak, P., *Projective Unification in Modal Logic*, **Logic Journal of the IGPL** 20(2012) No.1, 121–153.
6. Ghilardi S., *Unification through Projectivity*, **Journal of Symbolic Computation** 7 (1997), 733–752.

7. Ghilardi S., *Unification in Intuitionistic Logic*, **Journal of Symbolic Logic** 64(2) (1999), 859–880.
8. Ghilardi S.,*Best Solving Modal Equations*, **Annals of Pure and Applied Logic** 102 (2000), 183–198.
9. Goudsmit J., Iemhoff R., *On Unification and Admissible Rules in Gabbay-de Jongh Logics*, **Annals of Pure and Applied Logic** 165(2) (2014), 652–672.
10. Minari P. , Wroński A., *The property (HD) in Intuitionistic Logic. A Partial Solution of a Problem of H. Ono*, **Reports on Mathematical Logic** 22 (1988), 21–25.
11. Pogorzelski, W.A., Prucnal, T. *Structural Completeness of the First-order Predicate Calculus*, **Zeitschrift für Mathematische Logik und Grundlagen der Mathematik**, 21 (1975), 315-320.
12. Wroński A., *Transparent Verifiers in Intermediate Logics*, **Abstracts of the 54-th Conference in History of Mathematics**, Cracow (2008).