Dense completeness theorem for protoalgebraic logics*

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The paper [3] started a new approach to Abstract Algebraic Logic in which, instead of the usual equivalence-based classification of logical systems leading to the well-known Leibniz hierarchy of protoalgebraic logics (see [5]), we proposed an alternative setting based on implication connectives. By studying the properties of weak p-implications (understood as generalized connectives defined by sets of formulae $\Rightarrow$ in two variables and, possibly, with parameters), we obtained a refinement of the Leibniz hierarchy. Moreover, it was shown that they define an order relation in the semantical counterpart of these logics, i.e. in their reduced matrix models. This yielded a natural definition of *semilinear weak p-implications* as those that endow the logic with a complete semantics of *linearly ordered* matrix models. In particular examples studied in the literature of many-valued logics such completeness is often refined to particular kinds of linearly ordered models, typically those where the order is dense (see e.g. [2]). The aim of this contribution is to provide a general characterization of the logics that admit such completeness theorem with respect to their densely ordered linear matrix models.

Let L be a logic with a weak p-implication $\Rightarrow$ (i.e. an arbitrary protoalgebraic logic) in a propositional language $\mathcal{L}$. Assume that the cardinality of both the set of variables and the set of all formulae is the same and denote it as κ. Let $\boldsymbol{A}$ be an $\mathcal{L}$-algebra. A filter $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is $\Rightarrow$*-dense* if for every $a, b \in A$:

- $a \leq^{\Rightarrow}_{\mathbf{A}} b$ or $b \leq^{\Rightarrow}_{\mathbf{A}} a$,[3]
- if $a <^{\Rightarrow}_{\mathbf{A}} b$ there is $c \in A$ such that $a <^{\Rightarrow}_{\mathbf{A}} c$ and $c <^{\Rightarrow}_{\mathbf{A}} b$.

A matrix $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ is called a *dense linear matrix* w.r.t. $\Rightarrow$, $\mathbf{A} \in \mathbf{MOD}^{\delta}_{\Rightarrow}(\mathrm{L})$ in symbols, if it is reduced and F is $\Rightarrow$-dense (equivalently: if $\leq^{\Rightarrow}_{\mathbf{A}}$ is a dense linear order). Finally we say that L enjoys the *dense completeness* w.r.t. $\Rightarrow$ whenever $\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}^{\delta}_{\Rightarrow}(\mathrm{L})}$.

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[3] We define $a \leq^{\Rightarrow}_{\mathbf{A}} b$ iff $\chi^{\boldsymbol{A}}(a, b, c_1, \dots, c_n) \in F$ for each $\chi(p, q, r_1, \dots, r_n) \in \Rightarrow$ and $r_1, \dots, r_n \in A$. We also define $a <^{\Rightarrow}_{\mathbf{A}} b$ as $a \leq^{\Rightarrow}_{\mathbf{A}} b$ and $b \not\leq^{\Rightarrow}_{\mathbf{A}} a$.

There is a strong relation between dense completeness and properties of dense filters. One would expect that, in analogy with other completeness properties, given a set $\Gamma \cup \{\varphi\}$ of formulae such that $\Gamma \nvdash_{\mathrm{L}} \varphi$ there should be a $\Rightarrow$-dense theory $T \supseteq \Gamma$ such that $T \nvdash_{\mathrm{L}} \varphi$. However, perhaps unsurprisingly as density is not a universal property, this straightforward extension property will fail in general, as shown by the following example.

Example 1. Let us take the Łukasiewicz infinitely-valued logic Ł, which is well-know to enjoy the dense completeness w.r.t. its implication $\to$, and consider the theory T generated by the set Var. Clearly $T \nvdash_{Ł} \bot$, for each formula φ we have $T \vdash_{Ł} \varphi$ or $T \vdash_{Ł} \neg\varphi$, and hence it is maximally consistent and not $\to$-dense ($\bot <_{\langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle} \top$ but there is no formula φ such that $\bot <_{\langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle} \varphi <_{\langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle} \top$). Therefore T *cannot* be a extended into any $\to$-dense theory.

Theorem 1. *Let* L *be a logic with a weak p-implication* $\Rightarrow$. *Then the following are equivalent:*

- L *has the dense complete w.r.t.* $\Rightarrow$.
- L *has the* dense extension property w.r.t. $\Rightarrow$, *i.e. for every set* $\Gamma \cup \{\varphi\}$ *of formulae such that* $\Gamma \nvdash_{\mathrm{L}} \varphi$ *and there are* κ *many variables not occurring in* Γ, *there is an* $\Rightarrow$*-dense theory* $T \supseteq \Gamma$ *such that* $T \nvdash_{\mathrm{L}} \varphi$.

Another important property of implication, semilinearity, has been captured in terms of a metarule in [3] and, similarly, we have studied metarules for useful properties of disjunctions in [4]. Therefore, it is also to be expected that dense completeness should be equivalent to some suitable metarule. Indeed, the *density property* DP, appeared originally in [7] in a much more specific context and later was generalized to a wide class of fuzzy logics in [6]. Later this metarule has been studied in [1] in a very general context of hypersequent calculi (in a level of generality incomparable with ours). Such metarule needs a notion of a *fresh/unused* variable, which causes some expected issues with structurality which force us to restrict, in the next theorem, to finitary logics with a finite parameter-free implication; we also need to use the general notion of disjunction as studied in [4].

Theorem 2. *Let* L *be a* finitary *logic with a finite weak implication* $\Rightarrow$ *and a finite disjunction* ∇. *Then the following are equivalent:*

- L *is dense complete w.r.t.* $\Rightarrow$.
- L *has the dense extension property w.r.t.* $\Rightarrow$.
- L *has the* semilinearity *and* density *properties w.r.t.* $\Rightarrow$ *and* ∇, *i.e,*

$$\frac{\Gamma, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \Rightarrow \varphi \vdash_{\mathrm{L}} \chi}{\Gamma \vdash_{\mathrm{L}} \chi} \qquad\qquad \frac{\Gamma \vdash_{\mathrm{L}} (\varphi \Rightarrow p)\nabla(p \Rightarrow \psi)\nabla\chi}{\Gamma \vdash_{\mathrm{L}} (\varphi \Rightarrow \psi)\nabla\chi}$$

for any $\Gamma \cup \{\varphi, \psi, \chi\}$ *and any variable* p *not occurring in* $\Gamma \cup \{\varphi, \psi, \chi\}$.

Corollary 1. *Let* L *be a finitary logic with a finite weak semilinear implication* $\Rightarrow$ *and a finite disjunction* ∇. *Then* L *enjoys the dense completeness iff* L *equals the intersection of all its (finitary) extensions satisfying the density property.*

This corollary gives some insight into an approach used in the fuzzy logic literature to prove dense completeness (e.g. in [6] for the Uninorm Logic UL). Indeed, in this approach one starts from a suitable proof-theoretic description of a logic L, which then is extended into a proof-system for the intersection of all extensions of L satisfying the density property just by adding density property as a rule (in the proof-theoretic sense). This rule is then shown to be eliminable (using analogs of the well-known cut-elimination techniques), i.e., the condition is met and hence the original logic has dense completeness (of course, our general theory is not helpful in this last step, because here one needs to use specific properties of the logic in question).

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