Constructing representations of ordered monoids

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A *sup-lattice* is a complete lattice, a *morphism of sup-lattices* is a mapping preserving arbitrary suprema. A *Quantale* is a sup-lattice with an associative multiplication which distrubutes over arbitrary suprema. A *quantale module* is then a sup-lattice on which the quantale "naturally" acts. All sup-lattice endomorphisms of sup-lattice S form a quantale, denoted by $\mathcal{Q}(S)$. If Q is a quantale and M its left module, then the action provides morphism of quantales $Q \to \mathcal{Q}(M)$, called *representation* of Q. Conversely, a representation $Q \to \mathcal{Q}(S)$ endows S with a Q-module structure.

Quantales are obviously *residuated lattices*, i.e. they admit arrow operations $\leftarrow, \to$ given by

$$a \leq b \leftarrow c \;\Leftrightarrow\; a \cdot b \leq c \;\Leftrightarrow b \leq c \to a.$$

Categorically speaking, the arrows are the right adjoints to the left/right action, respectively. Thus, one could also consider the arrows for the module actions. Similarly to ring theory, it is also possible to derive matrix calculus for quantales (modules). (Since we have all suprema, includuing infinitesimal ones, the matrices here are mappings $I \times J \to Q$ for arbitrary index sets I, J.)

In [1] I showed that every left Q-module of a unital quantale Q can be constructed as a dual sup-lattice of $Q^I \to P = \{a \to P \mid a \in Q^I\}$ where a runs through all row vectors of type I and P is a suitable matrix over Q of type I, J.

It is well known that if K is another quantale and $f : K \to Q$ a quantale morphism, then every Q-module is also a K-module by "restricting scalars". We can use this fact also for maps $f : K \to Q$ which are not quantale morphisms (and K is not more a quantale) but idempotent semiring morphisms, partially or totally ordered monoid morphisms, etc., that is, they preserve the multiplication but not all suprema (e.g. finite suprema or just the preorder). In particular, we can consider embeddings $f : K \to Q$ where such a weaker structure K is extended to a quantale via an appropriate ideal completion. For example, if K is a preordered monoid, then we take all down-sets as Q, if K is an idempotent semiring (i.e. semilattice with multiplication respecting finite joins), then elements of Q are the lattice ideals.

Putting together the two ideas we get easily a method how to classify modules/representations for a large range of "quantale-like" structures. In particular we are interested in representations of totally ordered monoids (cf. [2]).

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References

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