Ideals and involutive filters in residuated lattices

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A *bounded integral residuated lattice* (= *residuated lattice*, for short) is an algebra $M = (M;\ \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 2, 0, 0, \rangle$ such that (i) $(M;\ \odot, 1)$ is a (non-necessarily commutative) monoid; (ii) $(M;\ \vee, \wedge, 0, 1)$ is a bounded lattice; (iii) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$ for any x, y $z \in M$. Put $x^- = x \rightarrow 0$, $x^\sim = x \rightsquigarrow 0$ for each $x \in M$. A residuated lattice is called (i) *good* if it satisfies the identity $x^{-\sim} = x^{\sim-}$; (ii) *involutive* if it satisfies the identities $x^{-\sim} = x = x^{\sim-}$.

Residuated lattices form a large class of algebras containing certain classes of algebras behind many-valued and fuzzy logics (commutative or non-commutative), e.g., MV-algebras, BL-algebras, MTL-algebras ([1], [7], [4]), and their non-commutative variants GMV-algebras (= pseudo-MV-algebras), pseudo-BL-algebras, pseudo-MTL-algebras ([6], [10], [2], [3], [5]).

In our talk we will deal with algebraic structure properties of residuated lattices.

It is well known that congruences on any residuated lattice are in a one-to-one correspondence with their normal filters. GMV-algebras (and, in particular, MV-algebras) can be considered, from the point of view of residuated lattices, as residuated lattices which satisfy identities of divisibility and pre-linearity and are involutive. If M is such a GMV-algebra, one can define the binary operation $\oplus$ such that $x \oplus y = (x^- \odot y^-)^\sim = (x^\sim \odot y^\sim)^-$ for any $x, y \in M$. Then we also have $x \odot y = (x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$, i.e. the operations $\odot$ and $\oplus$ are mutually dual. Using the operation $\oplus$, one can define the notion of an ideal (and a normal ideal), which is dual to the notion of a filter (and a normal filter). That means, congruences of GMV-algebras are also in a one-to-one connection with normal ideals. But in general, a dual operation to the multiplication in residuated lattices does not exist. Consequently a notion of "the (precise) dual to a filter" does not exist as well. Nevertheless, in [9] a kind of an ideal of a BL-algebra (which need not be an MV-algebra) has been introduced and it was shown that such ideals are very useful in the study of structure properties of BL-algebras. Among others, it is possible to define quotient BL-algebras not only using filters but in particular cases also using ideals. Namely, quotient BL-algebras induced by ideals are in fact MV-algebras.

We introduce the notion of an ideal of general residuated lattices (which need not be commutative). For this we define two binary operations $\oslash$ and $\otimes$ called the left and right additions such that: $x \oslash y := y^- \rightsquigarrow x$ and $x \otimes y := x^\sim \to y$. Then $\emptyset \neq I \subseteq M$ is called an *ideal* of M if (1) $x, y \in I \implies x \oslash y \in I$; (1') $x, y \in I \implies x \otimes y \in I$; (2) $x \in I,\ z \in M,\ z \leq x \implies z \in I$. We show that every ideal I of a residuated lattice M induces a congruence θ_I on M and that the quotient residuated lattice M/θ_I is involutive.

Let M be a residuated lattice. Then we say that M *satisfies the Glivenko property* (GP), if for any $x, y \in M$, $(x \to y)^{-\sim} = x \to y^{-\sim}$, $(x \rightsquigarrow y)^{\sim-} = x \rightsquigarrow y^{\sim-}$. If M is a residuated lattice then we denote $D(M) := \{x \in M : x^{-\sim} = 1 = x^{\sim-}\}$, the set of *dense* elements of M. We say that a normal filter F of M is *involutive* if the quotient residuated lattice M/F is involutive. We show that if M is a good residuated lattice satisfying (GP) then the involutive filters of M are exactly all normal filters of M containing $D(M)$. We describe connections between ideals and normal filters of M.

Let M be a residuated lattice. Then we get: a) If I is an ideal of M then I is the 0-class in M/θ_I. b) If F is a normal filter of M, then the class $0/F$ is an ideal of M.

Moreover we prove that if I is an ideal of a pseudo BL-algebra and $F = F_I = 1/\theta_I$, then F is an involutive normal filter of M.

It is known that the variety of residuated lattices is 1-regular, but not regular [8]. Hence every congruence θ on a residuated lattice M is determined uniquely by the filter $F_\theta = 1/\theta$, but other classes in M/θ can be at the same time also classes in different congruences on M. Nevertheless, we prove that if M is an arbitrary pseudo BL-algebra then there is a one-to-one correspondence between ideals and involutive normal filters of M.

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