On decidability of some classes of Stone algebras

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A typical (nontrivial) first order theory is undecidable. According to an early result of Tarski [15], the theory of Boolean algebras is one of the lucky exceptions. This was extended by Ershov to the decidability (of the theory) of relatively complemented distributive lattices as well as to Boolean algebras with a distinguished ultrafilter or prime ideal in [6], and to Post algebras in [7]. As shown by Rabin [13], even the second order theory of Boolean algebras with quantification over ideals is still decidable, implying the decidability of the first order theory of Boolean algebras with a sequence of distinguished ideals. On the other hand, several seemingly moderate generalizations of Boolean algebras are already undecidable. They include (bounded) distributive lattices (Grzegorczyk [8]), Boolean pairs, i.e., Boolean algebras with a distinguished subalgebra (Rubin [14]), all varieties of Heyting algebras properly extending the variety of Boolean algebras (Burris [3]), etc.

In our contribution we pursue examining the borderline between decidability and undecidability in the close vicinity of Boolean algebras. Iterating Katriňák's version of the Chen-Grätzer triple construction applied consecutively to n Boolean algebras (see [10], [4], [5]) we introduce the finitely axiomatizable classes $\mathbf{SA}_n$ of n-th degree Stone algebras as follows. Given Boolean algebras $B_1, B_2, \dots, B_n$ and their homomorphisms $h_i\, B_i \to B_{i+1}$ $(1 \le i < n)$, referred to as the *structure maps*, we take the *P-lattice*

$$B_1 \rtimes_{h_1} B_2 \rtimes_{h_2} \dots \rtimes_{h_{n-1}} B_n = \{(b_1, b_2, \dots, b_n) \in B_1 \times B_2 \times \dots \times B_n \mid h_1(b_1) \ge b_2,\ h_2(b_2) \ge b_3, \dots, h_{n-1}(b_{n-1}) \ge b_n\},$$

regarded as a $(0,1)$-sublattice of the direct product $B_1 \times B_2 \times \dots \times B_n$, with pseudocomplement

$$(b_1, b_2, \dots, b_n)^* = (b_1^*, h_1(b_1^*), \dots, (h_{n-1} \circ \dots \circ h_1)(b_1^*)).$$

Next we introduce two finitely axiomatizable subclasses $\mathbf{SA}^{\mathrm{i}}$ and $\mathbf{SA}^{\mathrm{s}}$ of the class $\mathbf{SA}_n$, with all the structure maps h_i in their P-lattice representation injective or surjective, respectively. Then the class of all Post algebras of degree n is definitionally equivalent to the intersection $\mathbf{SA}^{\mathrm{i}}_n \cap \mathbf{SA}^{\mathrm{s}}_n$ (cf. Katriňák-Mitschke [11], Balbes-Dwinger [2]). Taking the liberty of confusing first order languages, the class $\mathbf{SAD}_n$ of all n-th degree Stone algebras which are dually pseudocomplemented and form a dual Stone algebra under the operation of dual pseudocomplement satisfies the inclusions $\mathbf{PA}_n \subseteq \mathbf{SAD}_n \subseteq \mathbf{SA}^{\mathrm{s}}_n$.

Building on Rubin's undecidability proof of the class of Boolean pairs [14] we show that already the class $\mathbf{SA}^{\mathrm{i}}_2$ of all Stone algebras with Boolean dense elements set and injective structure map h_1 is hereditarily undecidable, hence all the classes $\mathbf{SA}^{\mathrm{i}}_n$ are undecidable for $n \geq 2$, too. The same is true for the classes $\mathbf{SA}_n$ and the class of all *Gödel algebras*, i.e., Heyting algebras satisfying $(x \to y) \vee (y \to x) = 1$.

On the other hand, using Rabin's method of interpretation (semantic embedding) from [12] and his above mentioned result from [13] we show that all the classes $\mathbf{SA}^{\mathrm{s}}_n$ are decidable. As a consequence we obtain the decidability of the classes $\mathbf{SAD}_n$, as well as another proof of Ershov's decidability result for the classes $\mathbf{PA}_n$. Finally, from a result of K. and P. Idziak [9], characterizing varieties of Heyting algebras with decidable first order theory of their finite members, it follows that the classes of all finite algebras in $\mathbf{SA}_n$ are decidable for each n.

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