Between axioms and structural rules in the display calculus

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The display calculus (of a logic) can be viewed as the proof-theoretic face of the logic's algebraic semantics, enabling us to investigate the logic using the tools of proof-theory. Here we will consider how to obtain analytic display calculi for axiomatic extensions of a logic (using substructural logics as an example), and then discuss how the issue of conservativity between the logic of the display calculus and certain sublogics can be investigated directly in this setting.

In order to study a logic from a proof-theoretical perspective, it is necessary to first obtain an analytic proof-calculus for the logic. By an analytic proof-calculus we mean that the calculus has the subformula property i.e. any proof in the calculus (of some formula ψ, say) uses only those formulae that are subformulae of ψ. The point is that the proofs then have nice structural properties, facilitating further analysis. Despite much work on this topic since Gentzen's [6] seminal work on analytic sequent calculi for intuitionistic and classical first-order logics, many logics still do not have an analytic proof-calculus. Even when an analytic calculus $\mathcal{C}$ is known for some logic L, it is often unclear how to obtain an analytic calculus for an axiomatic extension L+Ax. The situation is particularly vexing since many logics are constructed as axiomatic extensions of existing logics.

Belnap's Display Calculus [1] is a proof-theoretic formalism which generalises Gentzen's sequent calculus and is suitable for presenting logics whose logical operators are residuated. Indeed, the display calculus can be viewed as the proof-theoretic face of the algebraic semantics of a fully residuated logic. In particular, the residuation of the logical operators corresponds to a powerful structural property from the proof-theoretic perspective: the display property. Another attractive proof-theoretic feature is the general cut-elimination theorem which applies whenever the rules of the display calculus obey certain easy-to-verify conditions. Indeed, the formalism has been applied to give analytic calculi for many different families of logics including substructural logics, tense logics and bunched implication logics.

Here we address the question of computing analytic display calculi for axiomatic extensions of a logic. In particular, we identify a class of axioms such that every axiomatic extension using these axioms has an analytic calculus. Previous work on this topic has focussed on the display calculus [7] for tense logic and the hypersequent calculus [2] for Full Lambek logic. In contrast, our sufficient conditions are stated abstractly rather than for concrete calculi. Our results [3, 4] apply to many well-known display calculi, immediately yielding analytic calculi for the axiomatic extensions of the corresponding base logics. The set of axioms that we treat with our procedure is determined by the invertible rules of the base calculus. We can also show that it is decidable if an axiom belongs to this set or not.

Next we consider the reverse direction. We show that under certain additional conditions, every structural rule satisfying the conditions for cut-elimination is equivalent to an axiom from the set we identified before. In this way we give a full characterisation of this set of structural rules. Kracht [7] has shown a similar characterisation to ours for the concrete case of tense logic. Kracht's result can be obtained as a special case of the result presented here.

We conclude by discussing the vexing issue of conservativity that arises naturally when we wish to consider sublogics in a restricted language. Specifically, suppose that the logic L is defined in the language $\mathcal{L}$, and the logic $L' \subset L$ is defined in the language $\mathcal{L}'$. Then L is said to be *conservative* over L' if every theorem of L in the language $\mathcal{L}'$ is a theorem of L'. Suppose that $\mathcal{C}$ is an analytic display calculus for L and suppose that δ is a derivation in $\mathcal{C}$ of a formula ψ in the language $\mathcal{L}'$. Now δ can be seen viewed in the usual way as a directed tree with root ψ (the directed edges correspond to the rules of the calculus and the nodes correspond to sequents). Notice that δ does *not* witness a proof of ψ in the sublogic L' because certain nodes in δ may not be even interpretable in $\mathcal{L}'$. However, if we can extract a new tree from δ whose nodes *are* interpretable in $\mathcal{L}'$, then conservativity amounts to showing that each edge in the new tree corresponds to a valid inference in L'. The point is that the interesting theoretical result of conservativity can be expressed proof-theoretically as a transformation on δ. Moreover, the conservativity result then yields an analytic calculus for L'. Conservativity [5] of bi-intuitionistic linear logic over full-intuitionistic linear logic has already been shown in this this way. The analytic display calculi obtained above thus pave the way for the study of conservativity for a large class of logics.

References

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