Tense operators in logics without negation

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Abstract. For distributive lattices, the so-called positive modal algebra was already introduced by Dunn. He presented, using this notion, a canonical model and completeness theorem for positive modal logic.
Following Dunn's approach, but entirely by purely algebraic means, we introduce tense distributive algebras, i.e., bounded distributive lattices equipped with tense operators G, P, H and F. Tense operators express the quantifiers "it is always going to be the case that" and "it has always been the case that" and hence enable us to express the dimension of time in a logic without negation or any kind of implication.
Since distributive lattices have a nice theory of prime filters, we can generalize several approaches presented for tense Boolean algebras where a theory of ultrafilters is used. In the case when the pairs of tense operators (G,F) and (H,P) form the so-called MN-positive pairs, the representation problem of tense distributive algebras is solved.

Main results

Propositional logics, both clasical and non-classical, usually do not incorporate the dimension of time. However, already Aristotle mentioned that time plays an important role in evaluation of true values of propositions. This motivated a lot of authors to investigate the so-called Temporal logic, i.e., the logic where time is considered as a variable of the propositional formula.

In what follows, we will submit algebraic tools for the axiomatization of tense operators on distributive lattices. The first step to the axiomatization of tense operators then will be a construction using the time frame (T,R). The second step is a solution of the representation problem in a case when the pairs of tense operators (G,F) and (H,P) form the so-called MN-positive pairs.

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Definition 1. By the *tense distributive algebra* is meant an algebra $(A;\vee,\wedge,0,1,G,P,H,F)$ such that $\mathbf{A}=(A;\vee,\wedge,0,1)$ is a bounded distributive lattice with an induced order $\leq$, (P,G) and (F,G) are Galois connections on A such that, for all $p,q\in A$,

$$G(p)\leq G(0)\vee F(p) \quad \text{and} \quad F(1)\wedge G(p)\leq F(p),$$
$$H(q)\leq H(0)\vee P(q) \quad \text{and} \quad P(1)\wedge H(q)\leq P(q).$$

G,P,H and F are called *tense operators* on the tense distributive algebra $(\mathbf{A};G,P,H,F)$.

Let $(\mathbf{A}_1;G_1,P_1,H_1,F_1)$ and $(\mathbf{A}_2;G_2,P_2,H_2,F_2)$ be tense distributive algebras. A *morphism of tense distributive algebras* is a morphism of distributive lattices $f:A_1\to A_2$ which simultaneously commutes with the respective tense operators.

A *time frame* is a pair (T,R) where T is a non-empty set and $R\subseteq T\times T$.

Theorem 1. *Let $\mathbf{L}=(L;\vee,\wedge,0,1)$ be a finite distributive lattice and let (T,R) be a time frame. Define $\widehat{G},\widehat{P},\widehat{H},\widehat{F}$ of L^T into itself as follows: For all $p\in L^T$ and all $s\in T$,*

$$\widehat{G}(p)(s)=\bigwedge\nolimits_L\{p(t)\mid sRt\}$$
$$\widehat{F}(p)(s)=\bigvee\nolimits_L\{p(t)\mid sRt\}$$

and, for all $q\in L^T$ and all $t\in T$,

$$\begin{array}{l}\widehat{P}(q)(t)=\bigvee_L\{q(s)\mid sRt\}\\ \widehat{H}(q)(t)=\bigwedge_L\{q(s)\mid sRt\}\end{array}.$$

Then $(\mathbf{L}^T;\widehat{G},\widehat{P},\widehat{H},\widehat{F})$ is a tense complete distributive algebra such that, for all $p_1,p_2,q_1,q_2\in L^T$,

$$\widehat{G}(p_1\vee p_2)\leq\widehat{G}(p_1)\vee\widehat{F}(p_2) \quad \textit{and} \quad \widehat{F}(p_1)\wedge\widehat{G}(p_2)\leq\widehat{F}(p_1\wedge p_2), \tag{MNP1}$$
$$\widehat{H}(q_1\vee q_2)\leq\widehat{H}(q_1)\vee\widehat{P}(q_2) \quad \textit{and} \quad \widehat{P}(q_1)\wedge\widehat{H}(q_2)\leq\widehat{P}(q_1\wedge q_2). \tag{MNP2}$$

Recall that, for any bounded distributive lattice $\mathbf{A}=(A;\vee,\wedge,0,1)$, we have a full set $T_{\mathbf{A}}^{\text{dist}}$ of morphisms of bounded lattices into the two-element bounded distributive lattice $\mathbf{2}=(\{0,1\};\vee,\wedge,0,1)$.

The following result is well known.

Proposition 1. *Let $\mathbf{A}=(A;\vee,\wedge,0,1)$ be a distributive lattice. Then the map $i_{\mathbf{A}}:A\to 2^{T_{\mathbf{A}}^{dist}}$ given by $i_{\mathbf{A}}(a)(s)=s(a)$ for all $a\in A$ and all $s\in T_{\mathbf{A}}^{dist}$ is an order reflecting morphism of bounded lattices such that $i_{\mathbf{A}}(A)$ is a sub-distributive lattice of $\mathbf{2}^{T_{\mathbf{A}}^{dist}}$.*

Now we can establish our main result.

Theorem 2. (Representation theorem for tense distributive algebras) *Let $(\mathbf{A};G,P,H,F)$ be a tense distributive algebra,*

$$\begin{aligned}R_{G,H}=\{(s,t)\in T_{\mathbf{A}}^{dist}\times T_{\mathbf{A}}^{dist}\mid{}&(\forall b\in A)(s(G(b))\leq t(b))\ \textit{and}\\ &(\forall b\in A)(t(H(b))\leq s(b))\}.\end{aligned}$$

If (G,F) and (H,P) are positive MN-pairs, i.e., they satisfy conditions (MNP1) *and* (MNP2) *from Theorem 1 then the map $i_{\mathbf{A}}$ is an order reflecting morphism of tense distributive algebras into the powerset tense distributive algebra $(\mathbf{2}^{T_{\mathbf{A}}^{dist}};\widehat{G},\widehat{P},\widehat{H},\widehat{F})$ given by the time frame $(T_{\mathbf{A}}^{dist},R_{G,H})$.*

Conclusion and future work

We have developed a construction of a tense distributive algebra $(\mathbf{L}^T; \widehat{G}, \widehat{P}, \widehat{H}, \widehat{F})$ by means of a time frame (T,R) and a finite distributive lattice $\mathbf{L}$ such that the pairs $(\widehat{G}, \widehat{F})$ and $(\widehat{H}, \widehat{P})$ are positive MN-pairs. Conversely, given a tense distributive algebra $(\mathbf{A}; G, P, H, F)$ such that (G,F) and (H,P) are positive MN-pairs we can embed it into the powerset tense distributive algebra $(\mathbf{2}^{T_{\mathbf{A}}^{\mathrm{dist}}}; \widehat{G}, \widehat{P}, \widehat{H}, \widehat{F})$ constructed by means of the time frame $(T_{\mathbf{A}}^{\mathrm{dist}}, R_{G,H})$.

In [2], similar results were obtained for positive logics with adjoint modalities P_A and G_A, $A \in \mathcal{A}$ where $\mathcal{A}$ is a set of agents and our adjunction (P_A, G_A) has the following interpretation: $P_A(m)$ has to be interpreted as “agent A's uncertainty about a proposition m” and $G_A(m)$ has to be interpreted as “agent A's uncertainty about a proposition m”. To obtain a representation of their positive logics with adjoint modalities, Dyckhoff and Sadrzadeh use the so-called *multi-modal Kripke frame* which is a tuple $(W, \leq, (R_A)_{A\in\mathcal{A}}, (R_A^{-1})_{A\in\mathcal{A}})$ where W is a non-empty set, $\leq$ is a partial order on W, each R_A is a binary relation on W and R_A^{-1} is its inverse such that

$$\leq \circ R_A^{-1} \circ \leq \subseteq R_A^{-1} \quad \text{and} \quad \geq \circ R_A \circ \geq \subseteq R_A.$$

In our case, we have only two agents, say 1 and 2, and we put, due to Theorem 2, $W = T_{\mathbf{A}}^{\mathrm{dist}}$, $R_1 = R_{G,H}$ and $R_2 = R_{H,G} = R_{G,H}^{-1}$ and the partial order on our set W is trivial, i.e., the identity relation. Our machinery is based along the lines of [1], that is by using MN-positive pairs to build our time frame. Hence we can forget about the partial order. We believe that we will be able to combine both approaches to provide a new representation theorem for a variant of positive logics with adjoint modalities.

In [3] and [4], the authors construct intuitionistic analogues to classical tense logic using also the (intuitionistic) connectives $\to$ and $\neg$ which we omit in our approach.

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