Open Set Lattices of the Spaces of Minimal Prime Elements of Multiplicative Lattices

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Given a commutative ring R, the set $Spec(R)$ of all prime ideals of R equipped with the hull-topology is called the prime spectrum of R. In this abstract, the subspace of $Spec(R)$ of all maximal ideals of R is denoted by $MaxSpec(R)$ and the subspace of all minimal prime ideals of R is denoted by $MinSpec(R)$. One remarkable result in commutative ring theory is that the open set lattice $\mathcal{O}(Spec(R))$ of $Spec(R)$ is (order) isomorphic to the lattice of all radical ideals of R. For the subspace $MinSpec(R)$ of minimal prime ideals, some work had been done by N.K. Thakare and S.K. Nimbhorkar for a ring without nilpotent elements [14]. In [14], the authors showed if R is a ring with identity and without nilpotent elements, then an ideal is normal if and only if I is the intersection of all minimal prime ideals containing I (Corollary 3.12 of [14]). This could have established an isomorphism between the open set lattice of $MinSpec(R)$ of R and the lattice of all normal ideals if it is valid. Unfortunately, the key step (Corollary 3.12, [14]) in their proof made use of the following assumption which need not necessarily be true: the annihilator of a minimal prime ideal of a ring R is non-zero.

One natural abstraction of the set $Idl(R)$ of ideals of a commutative ring R is the multiplicative lattice first systematically investigated by R.P. Dilworth. One sustainable research topic on abstract ideal theory is to extend the results on ideals of commutative rings to multiplicative lattices. One such type of problem is: given a multiplicative lattice L and a subspace S of the spectrum $Spec(L)$ (all prime elements of L equipped with the hull topology) of L, can we find a subset of L that is order isomorphic to the open set lattice of S? In our earlier paper [13], we have considered this problem for two subspaces of $Spec(L)$: $Spec(L)$ itself and the subspace, $MaxSpec(L)$ of all maximal elements of L. In this abstract, we shall address the above problem for the subspace of all minimal prime elements of L. Our main result is that for certain multiplicative lattices L, the open set lattice of the subspace of all the minimal prime elements of L is isomorphic to the subset of L of all normal elements. Some links to the corresponding results for ideals of rings are given.

Note that for multiplicative lattices in which every element is compact, N.K. Thakare et. al. [15] had considered the elements which can be expressed as a meet of minimal prime elements. Since the multiplicative lattice $Idl(R)$ of all ideals of a commutative ring R may have non-compact elements, we need to consider a class of multiplicative lattices that includes at least $Idl(R)$ for commutative rings R.

For the case where L is a frame, J. Martinez et. al. [11] and T. Dube [5] had also considered the minimal spectra of L. P. Bhattacharjee [4] studied the frame of all minimal prime elements of an algebraic frame in which the meet of any two compact elements is a compact element.

Based on our main result in this paper, we also construct a counterexample to show that one of the main results in [14] is not always valid.

Definition 1. *A multiplicative lattice L is said to be coherent if L is algebraic, the top element, 1_L, of L is compact and for any compact elements a and b of L, ab is compact.*

L is said to be reduced if L has no non-zero nilpotent element.

Theorem 1. *For any prime element p of a reduced coherent multiplicative lattice L, the following statements are equivalent:*

(1) $C(p) = \{y \in K(L) : y \not\leq p\}$ is a maximal m-closed set of $K(L)$, the set of all compact elements of L.

(2) p is a minimal prime element.

(3) For any $a \in K(L) \cap \downarrow p$, there exists $b \in C(p)$ such that $ab = 0_L$.

For any element x of a multiplicative lattice L with bottom element 0_L, define $x^* = \bigvee\{y \in L : yx = 0_L\}$, called the annihilator of x.

Theorem 2. *Let L be a coherent multiplicative lattice. For any $a \in K(L)$ and $p \in Spec(L)$, the following statements are equivalent :*

(1) $a^* \leq p$.

(2) There exists some $q \in MinSpec(L)$ such that $q \leq p$ and $a \not\leq q$.

Lemma 1. *Let L be a reduced coherent multiplicative lattice. Then $x^* = \bigwedge\{p \in MinSpec(L) : x \not\leq p\}$ for any $x \in L$.*

An element x of a multiplicative lattice L is said to be a *normal* element if $x = x^{**}$.

Theorem 3. *Let L be a reduced coherent multiplicative lattice. The following statements are equivalent :*

(1) For all $p \in MinSpec(L)$, $p^* \neq 0_L$.

(2) For any minimal prime element p of L, $x \in L$ is such that $x \leq p$, then $x^* \not\leq p$.

(3) If $x \in L$ is a meet of some elements of $MinSpec(L)$, then x is normal.

Theorem 4. *Let L be a reduced coherent multiplicative lattice with all its minimal prime elements psuch that $p^* \neq 0_L$. Then the open set lattice of all minimal prime elements of L, equipped with the hull kernel topology, is isomorphic to the frame of all normal elements of L.*

Corollary 1. *Let R be a commutative ring with identity and with no nilpotent elements. If for any minimal prime ideal P of R, the annihilator of P, P^*, is not $\{0_R\}$, then the topology on the set of minimal prime ideals of R is isomorphic to the lattice of all normal ideals of R.*

The following is an example of a commutative ring R in which the annihilator of a minimal prime ideal of R is zero.

Example 1. Let R be the ring of all sequences from the ring $\mathbb{Z}/2\mathbb{Z}$ that are eventually constant, with the pointwise addition and multiplication. Then R is a commutative ring that has no nilpotent element. The subset $M_\infty = \{(a_n) \in \mathbb{R} : a_n = 0 \text{ holds eventually}\}$ is a minimal prime ideal of R. For any $x = (a_n) \neq 0_R \in M_\infty$, assume that $a_k \neq 0$, then $y = (0, 0, \cdots, 0, 1, 0, ...) \in M_\infty$ and $xy \neq 0_R$, where the kth component of y is 1 and others are 0. Thus the annihilator of M_∞ equals $\{0\}$.

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