A cut-free proof system for pseudo-transitive modal logics

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It is well known that Kripke semantics allows logicians to draw links between modal axioms and frame properties. For example, a modal logic obeys the axiom

$$\mathsf{4}\colon \Box A \supset \Box\Box A$$

if and only if all its models have a transitive frame.

More generally, a Kripke frame is said to be (m, n)-*transitive*, or simply *pseudo-transitive* (if n and m are clear from context), if its accessibility relation R satisfies $R^n \subseteq R^m$, where R^n is defined as R composed n times with itself[1].

We call K^m_n the logic obtained from the basic modal logic K by adding the axiom $\mathsf{4}^m_n$:

$$\mathsf{4}^m_n\colon\ \Box^m A \supset \Box^n A$$

where $\Box^n A$ is defined inductively as $\Box^0 A = A$ and $\Box^{n+1} A = \Box(\Box^n A)$, i.e. $\mathsf{4}^1_2$ is just the $\mathsf{4}$-axiom shown above. The theorems of K^m_n are exactly the modal formulas valid in all (m, n)-transitive frames. In this talk, we give a cut-free proof system for any such pseudo-transitive modal logic.

Many different proposals have been made to automate the design of cut-free proof systems in modal logic; for example, using logical rules in nested sequents [1, 7] or in the similar framework of tree-hypersequents [5], structural rules in hypersequents [3], or a mixture of structural and logical rules in nested sequents [4]. Recently, Fitting [2] introduced an alternative approach, using structural rules in *indexed nested sequents*, that might finally provide a general translation of axioms into rules.

A nested sequent is a multiset of formulas[2] and *bracketed* nested sequents. It can be seen as the generalisation of sequents to a structure of tree.

Example 1. The sequent $\Gamma = A, [B, [D]], [C]$ can be interpeted as the formula $A \vee \Box(B \vee \Box D) \vee \Box C$. The comma corresponds to $\vee$ and the brackets to $\Box$.

An indexed nested sequent is then a nested sequent where each node carries an *index*, that is a natural number. We write the index as superscript to the opening bracket.

Example 2. $\Gamma = A, [^1 B, [^2 D]], [^3 D, [^1 C, [^4 A]]]$

[1] The composition of two binary relations R, S on a set W is: $R \circ S = \{(w, v) \in W \times W \mid \exists u.\ wRu \wedge uSv\}$.

[2] We consider only formulas in negation normal form, generated from atoms $a, b, c, \ldots$, negated atoms $\bar{a}, \bar{b}, \bar{c}, \ldots$, via the usual connectives $\wedge, \vee, \Box, \Diamond$.

Since the same index can appear on different nodes, it generalises the structure of trees to more general graphs, depending on the conditions on the indexing. For example, if we disallow the same index to appear twice on a branch of the sequent tree, then the indexed sequent behaves like a directed acyclic graph (dag).

Like in the nested sequent framework, we use a notion of *context*, a sequent with one or several holes that can be filled with another sequent. In this indexed framework, each hole carries the same index as the bracket it appears in.

Example 3. For example $\Gamma\,{}^{1}\{\;\}\,{}^{2}\{\;\} = A, [^{1}B, [^{2}\{\;\}]], [^{3}D, [^{1}\{\;\}]]$ is a binary context. If we plug the sequents $\Delta = E$ and $\Sigma = F, [^{4}G]$ into its holes, we get:

$$\Gamma\,{}^{1}\{\Delta\}\,{}^{2}\{\Sigma\} = A, [^{1}B, [^{2}E]], [^{3}D, [^{1}F, [^{4}G]]] \quad .$$

The following rules are exactly the ones introduced in [1] for modal logic K, except that here we add the indexes, and demand that in the $\Box$-rule the index v does not appear in the conclusion:

$$\mathsf{id}\,\frac{}{\Gamma\{a,\bar{a}\}} \qquad \vee\,\frac{\Gamma\{A,B\}}{\Gamma\{A\vee B\}} \qquad \wedge\,\frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A\wedge B\}}$$
$$\Box\,\frac{\Gamma\{[^{v}A]\}}{\Gamma\{\Box A\}} \qquad \Diamond\,\frac{\Gamma\{\Diamond A, [^{u}A,\Delta]\}}{\Gamma\{\Diamond A, [^{u}\Delta]\}} \tag{1}$$

In the indexed setting, we also need the following two rules that allow to move formulas and brackets between nodes of the same index:

$$\mathsf{tp}\,\frac{\Gamma\,{}^{w}\{\emptyset\}\,{}^{w}\{A\}}{\Gamma\,{}^{w}\{A\}\,{}^{w}\{\emptyset\}} \qquad \mathsf{bc}\,\frac{\Gamma\,{}^{w}\{[^{u}\Delta]\}\,{}^{w}\{[^{u}\;]\}}{\Gamma\,{}^{w}\{[^{u}\Delta]\}\,{}^{w}\{\emptyset\}} \tag{2}$$

So far, we did not yet make actual use of the indexing. The point is that indexing allows us to construct a rule that corresponds to the axiom $\mathsf{4}^m_n$. It can be written as:

$$\dot{\mathsf{4}}^m_n\,\frac{\Gamma\{[^{v_m}\cdots[^{v_2}[^{w}\;]]], [^{u_n}\cdots[^{u_2}[^{w}\Delta_1],\Delta_2],\cdots\Delta_n]\}}{\Gamma\{[^{u_n}\cdots[^{u_2}[^{w}\Delta_1],\Delta_2],\cdots\Delta_n]\}} \tag{3}$$

where the indexes $v_2,\ldots,v_m$ must not appear in the conclusion[3].

The rules shown in (1), (2), and (3) together form *system* NK^m_n. Our main result is that it is sound and complete with respect to the logic K^m_n.

Theorem 1 (Soundness and Completeness). *A formula A is a theorem of K^m_n if and only if it is derivable in NK^m_n.*

The proof of soundness is rather straightforward. The completeness proof however uses a more involved syntactic cut-elimination procedure within indexed nested sequents.

[3] It is a special case of the rule $G^{k,l,m,n}$ introduced in [2].

Theorem 2 (Cut-Elimination). *If a sequent Γ is derivable in NK_n^m together with the* cut*-rule*

$$\mathsf{cut}\,\frac{\Gamma\{A\} \quad \Gamma\{\bar{A}\}}{\Gamma\{\emptyset\}}$$

then it is also derivable in NK_n^m without cut.

The standard cut-elimination procedure (permuting the cut up until the cut-formula is principal on both sides, and then reduce the cut-rank) is not aplicable in the presence of the tp-rule, that does not decrease the depth of the active formula from conclusion to premise. Therefore, we rather prove cut-elimination for a sytem $\ddot{\mathsf{N}}\mathsf{K}_n^m$ in which the tp-rule is admissible. This system is obtained from NK_n^m by replacing the identity rule and the $\Diamond$-rule as follows:

$$\ddot{\mathsf{id}}\,\frac{}{\Gamma^{\,u}\{a\}^{\,u}\{\bar{a}\}} \qquad \ddot{\Diamond}\,\frac{\Gamma^{\,u}\{\Diamond A\}^{\,u}\{[A,\Delta]\}}{\Gamma^{\,u}\{\Diamond A\}^{\,u}\{[\Delta]\}} \tag{4}$$

Since we can show that a sequent is provable in NK_n^m if and only if it is provable in $\ddot{\mathsf{N}}\mathsf{K}_n^m$, cut elimination for NK_n^m follows immediately from cut elimination for $\ddot{\mathsf{N}}\mathsf{K}_n^m$.

An advantage of cut-free sequent-like systems is their usability for proof search. We are currently working on decision procedures for pseudo-transitive modal logics[4] using the system presented here.

References

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[4] In [6] their decidability is mentioned as an open problem.