Isomorphism of knowledge bases: on the edge of logic and geometry

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In our work we study various equivalences of knowledge bases which allows us to compare different knowledge bases [1].

Our aim is to associate with a knowledge base certain logical and geometrical invariants and to study them from the positions of algebraic logic and logical geometry [2], [3], [4]. Such approach enables us to study logical notions and questions (which have a syntactical nature) using algebraic and geometrical structures and methods (via semantics) which are more visual and transparent.

The data that define a knowledge base $KB(H, \Psi, f)$ is a model $\mathcal{H} = (H, \Psi, f)$, where H is an algebra from a variety of algebras Θ, Ψ is a set of relation symbols and f is an interpretation of all symbols from Ψ in H. We will use the term "a knowledge base" instead of a more precise "a knowledge base model".

We shall emphasize that all of our notions are oriented towards an arbitrary variety of algebras Θ, therefore algebras, logic and geometry of knowledge bases are related to this variety.

The aim of the present talk is to discuss the following

Problem 1. What are the conditions which provide an isomorphism of two knowledge bases?

With this end, we describe an algebraic model of a knowledge base and formulate a system of logical and algebraic notions which provide sufficient conditions for knowledge bases isomorphism.

Speaking about knowledge we proceed from its representation in three components.

(1) *Description of knowledge* presents a syntactical component of knowledge. From an algebraic viewpoint, description of knowledge is a set of formulas T in the algebra of formulas $\Phi(X)$, $X = \{x_1, \dots, x_n\}$. We will not give precise definition of this algebra, for details see [1], [3], [5]. We only note that $\Phi(X)$ is a boolean algebra with operations $\wedge, \vee, \neg$. Moreover, it is a quantifier algebra, i.e., the existential quantifiers $\exists x_i$, for $x_i \in X$, are defined as unary operations. The signature of $\Phi(X)$ includes also infinitely many unary operations, denote them by one symbol s_*. The reason to introduce these operations is as follows. In logic and model theory we are working with an infinite set of variables, but for geometrical consideration we need a finite set of variables, in order to work with finite dimensional affine spaces.

(2) *Subject area of knowledge* is presented by a model (H, Ψ, f).

(3) *Content of knowledge* is a subset in H^n, where H^n is the Cartesian power of H. We present H^n as a set $Hom(W(X), H)$ of all homomorphisms from

$W(X)$ to H, where $W(X)$ is a free algebra in the variety Θ. Each content of knowledge A corresponds to the description of knowledge $T \subset \Phi(X)$, $|X| = n$. If we regard $Hom(W(X), H)$ as an affine space then this correspondence can be treated geometrically.

Let T be a set of formulas from $\Phi(X)$ and A a set of points in H^n. Define correspondence between a set of formulas and a set of point as follows (the definition of $LKer(\mu)$ see below):

$$T^L_{\mathcal{H}} = \{\mu \in H^n \mid T \subset LKer(\mu)\}, \quad A^L_{\mathcal{H}} = \bigcap_{\mu \in A} LKer(\mu).$$

In other words, $T^L_{\mathcal{H}}$ consists of all points satisfying all formulas from T, $A^L_{\mathcal{H}}$ is a set of all formulas which hold true on each point from A.

This correspondence is the Galois correspondence, that is, $T \subseteq T^L_{\mathcal{H}}$ and $A \subseteq A^L_{\mathcal{H}}$. The set $T^L_{\mathcal{H}}$ is called *definable set* presented by the set of formulas T. The set $A^L_{\mathcal{H}} \in \Phi(X)$ is called *$\mathcal{H}$-closed* (boolean) filter.

All definable sets in $Hom(W(X), H)$ form a lattice with operations union and intersection of definable sets. All $\mathcal{H}$-closed filters in $\Phi(X)$ also constitute a lattice (for the precise definition see [3]).

In order to describe the dynamic nature of a knowledge base, two categories are introduced: *the category of knowledge description $F_\Theta(\mathcal{H})$ and the category of knowledge content $LG_\Theta(\mathcal{H})$*. These categories are defined using the machinery of algebraic logic and logical geometry [1], [6].

Let a homomorphism $s : W(X) \to W(Y)$ of free algebras be given. An object $LG^X_\Theta(\mathcal{H})$ of the category $LG_\Theta(\mathcal{H})$ is the lattice of all definable sets in $Hom(W(X), H)$. Define a morphism in $LG_\Theta(\mathcal{H})$

$$[\widetilde{s}] : LG^Y_\Theta(\mathcal{H}) \to LG^X_\Theta(\mathcal{H}),$$

using the map $\widetilde{s}$ between definable sets in $Hom(W(Y), H)$ and $Hom(W(X), H)$, where $\widetilde{s}A = \{\mu s \mid \mu \in A\}$, $A \subset Hom(W(Y), H)$. If A is a definable set in $Hom(W(Y), H)$ then $[\widetilde{s}]A$ is the Galois closure of the set $\widetilde{s}A$, that is, $[\widetilde{s}]A = (\widetilde{s}A)^{LL}_{\mathcal{H}}$.

An object $F^X_\Theta(\mathcal{H})$ of the category $F_\Theta(\mathcal{H})$ is the lattice of $\mathcal{H}$-closed filters in $\Phi(X)$. We define morphism in $F_\Theta(\mathcal{H})$

$$[s_*] : F^X_\Theta(\mathcal{H}) \to F^Y_\Theta(\mathcal{H}),$$

using the map s_* between $\mathcal{H}$-closed filters in $\Phi(X)$ and $\Phi(Y)$ (for definition of s_* see [3], [5]). If T is an $\mathcal{H}$-closed filter in $F^Y_\Theta(\mathcal{H})$ then the corresponding $\mathcal{H}$-closed filter in $F^X_\Theta(\mathcal{H})$ is s_*T.

A knowledge base $KB(H, \Psi, f)$ is a triple $(F_\Theta(\mathcal{H}), LG_\Theta(\mathcal{H}), Ct_{\mathcal{H}})$, where $F_\Theta(\mathcal{H})$ is the category of knowledge description, $LG_\Theta(\mathcal{H})$ is the category of knowledge content, and

$$Ct_{\mathcal{H}} : F_\Theta(\mathcal{H}) \to LG_\Theta(\mathcal{H})$$

is a contravariant functor. The functor $Ct_{\mathcal{H}}$ transforms the knowledge description to the knowledge content and makes a knowledge base a dynamic object.

The definition of isomorphism of two knowledge bases $KB(H_1, \Psi, f_1)$ and $KB(H_2, \Psi, f_2)$ has in mind an isomorphism of categories of knowledge content $LG_\Theta(\mathcal{H}_1)$ and $LG_\Theta(\mathcal{H}_2)$, which implies isomorphism of categories of knowledge descriptions $F_\Theta(\mathcal{H}_1)$ and $F_\Theta(\mathcal{H}_2)$.

Relying on the model-theoretic notion of a type we present its logically-geometrical analogue, an LG-type. An *X-LG-type* of a point $\mu = (h_1, \ldots, h_n)$ is the set of all first order logic formulas in variables $X = \{x_1, \ldots, x_n\}$ (not necessarily free) written in a language L^* which hold true on the point μ:

$$LKer(\mu) = \{u(x_1, \ldots, x_n) \in L^* \mid H \models u(h_1, \ldots, h_n)\}.$$

The language L^* includes the symbol of operation s_* which we mentioned speaking about the algebra $\Phi(X)$, that is,

$$L^* = \{\wedge, \vee, \neg, \exists, s_*, \Psi\}.$$

Denote by $S^X(\mathcal{H})$ the set of all X-LG-types of the model $\mathcal{H}$. Models $\mathcal{H}_1 = (H_1, \Psi, f_1)$ and $\mathcal{H}_2 = (H_2, \Psi, f_2)$ are called *LG-isotypic*, if $S^X(\mathcal{H}_1) = S^X(\mathcal{H}_2)$, for each finite set of variables X. The following theorem takes place.

Theorem 1. *If models (H_1, Ψ, f_1) and (H_2, Ψ, f_2) are LG-isotypic then the corresponding knowledge bases $KB(H_1, \Psi, f_1)$ and $KB(H_2, \Psi, f_2)$ are isomorphic.*

There are some other approaches to compare two knowledge bases (see, for instance, [1]). One can speak about informational equivalence, elementary equivalence, logically-geometrical equivalence and others. Each of these approaches allows us to use specific methods and techniques which are more appropriate for studying given knowledge bases.

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