Uniform interpolation in modal logics

Rosalie Iemhoff*

Department of Philosophy, Utrecht University
Janskerkhof 13, The Netherlands
r.iemhoff@uu.nl
http://www.phil.uu.nl/~iemhoff

Interpolation has been studied in a variety of settings since William Craig proved that classical predicate logic has interpolation in 1957. Interpolation is considered by many to be a "good" property because it indicates a certain well-behavedness of the logic, vaguely reminiscent to analycity. In 1992 it was proved by Andrew Pitts that intuitionistic propositional logic IPC, which has interpolation, also satisfies the stronger property of *uniform interpolation*: given a formula φ and an atom p, there exist *uniform interpolants* $\forall p\varphi$ and $\exists p\varphi$ which are formulas (in the language of IPC) that do not contain p and such that for all ψ not containing p:

$$\vdash \varphi \to \psi \ \Leftrightarrow \vdash \exists p\varphi \to \psi \qquad \vdash \psi \to \varphi \ \Leftrightarrow \vdash \psi \to \forall p\varphi.$$

This is a strengething of interpolation in which the interpolant only depends on the premiss (in the case of $\exists$) or the conclusion (in the case of $\forall$) of the given implication. As the notation suggests, the fact that the uniform interpolants are definable in IPC also shows that the propositional quantifiers are definable in that logic.

Around the same time that Pitts' result appeared, [7] proved, by completely different methods, that the modal logic GL has uniform interpolation. Since then, uniform interpolation has been established for various other logics, including the modal logics K and KT [1, 8]. Intriguingly, the modal logics K4 and S4 do not have uniform interpolation [1, 3]. As there are only seven propositional intermediate logics with interpolation [5], the number of intermediate logics with uniform interpolation is necessarily bounded by that number. [4] showed that there are exactly that many.

Whereas in the presence of a decent analytic sequent calculus, proofs of interpolation are often relatively straightforward, proofs of uniform interpolation are in general quite complex. Moreover, it is less clear in how far, if at all, proof systems such as sequent calculi can be of help in establishing the property, as there are logics with analytic sequent calculi that have uniform interpolation (K and GL) as well as logics with analytic sequent calculi that do not (K4 and S4).

In this paper our aim is twofold: to develop a method to extract uniform interpolants from sequent calculi and to prove, using this method, that logics

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without uniform interpolation lack certain calculi. For both aims it holds that the more general the calculi we consider are, the stronger the result. In this paper we restrict ourselves to classical propositional modal logics, but the method applies to intermediate logics as well. For such logics the method is more complicated though, since $\exists$ is not expressible in terms of $\forall$, whereas in the classical case one can just take $\neg\forall p\neg$ for $\exists p$. We treat intermediate logics in a seperate paper.

Because of the way in which we construct uniform interpolants on the basis of calculi, we reach another goal as well. Namely that of providing a modular approach to uniform interpolation, meaning that the relation between a particular rule in a calculus and the property of uniform interpolation of the whole calculus is clearified. Our method is different from but inspired by Pitts' ingenious syntactic method. [1] used a similar method as Pitts to treat K, GL, KT and Grz. Most other proofs of uniform interpolation are of a semantical nature.

We isolate a certain type of propositional rules called *focussed rules* and a certain type of modal rules called *focussed modal rules* and prove that any logic with a terminating balanced sequent calculus consisting of focussed and focussed modal rules has uniform interpolation. Termination means that in no rule the premisses are more complex, in a certain ordering, than the conclusion. And a calculus is balanced if for certain combinations of left and right rules, either both rules belong to the calculus or both do not. This result then implies the well-known fact that classical propositional logic has uniform interpolation, and that so have K and KD. It also implies that K4 and S4 cannot have sequent calculi of the above kind. Although for S4 this might be easy to infer in another way, for K4 this seems to be a novel insight.

Furthermore, uniform interpolation is obtained for various other modal logics. The main interest in these results lies not so much in the logics involved, but rather in the illustration they provide of the flexibility of the method developed here. The calculi covered in this paper are not the only calculi to which our method applies, or so we conjecture. It seems likely that similar reasoning applies to other calculi for modal and intermediate logics. We chose, however, to first set up the general framework in this paper, mainly because we think it is of interest in itself and to seperate it from the complexeties that might be uncovered in applying it to other calculi than the ones treated here.

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