Complex algebras of tree-semilattices

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Complex algebras of relational structures play an important role in algebraic logic since they connect relational and algebraic semantics of (poly)modal logics. Moreover, several varieties of algebraic logic are generated by complex algebras, such as representable relation algebras (generated by complex algebras of Brandt groupoids) and modal algebras (generated by complex algebras of directed graphs).

Complex algebras are Boolean algebras with operators (BAOs), with the operators obtained by lifting the relations/operations of the structure to its powerset. Determining an axiomatization for the variety of BAOs generated by the complex algebras of a class of structures can be an interesting problem. A general approach is given by [2], where it is shown that for a recursively enumerably axiomatized class of algebras, the class of subalgebras of the complex algebras has a recursive axiomatization.

Here we consider the variety generated by complex algebras of semilattices. It has been studied in a manuscript [1] and some results about the larger variety generated by complex algebras of semigroups are in [3].

For a semilattice $(S, \cdot)$, let $\mathrm{Cm}(S) = (\mathcal{P}(S), \cup, \cap, -, \emptyset, S, \cdot)$ be the complex algebra, where for $x, y \in \mathcal{P}(S)$ we define $x \cdot y = \{r \cdot s \mid r \in x, s \in y\}$. Note that we do not distinguish notationally between the semilattice meet and its lifted version, and we usually write $x{\cdot}y$ simply as xy, and $xx = x^2$. Subalgebras of these BAOs are called *representable Boolean semilattices*, and the variety generated by all complex algebras of semilattices is denoted by RBSL. The larger variety BSL of *Boolean semilattices* is the class of algebras $(A, \vee, \wedge, -, 0, 1, \cdot)$ such that

- $(A, \vee, \wedge, -, 0, 1)$ is a Boolean algebra,
- $(A, \cdot)$ is a commutative semigroup, and
- $\cdot$ is a *square-increasing operator*: $x \leq x^2$, $x(y \vee z) = xy \vee xz$, and $x0 = 0$.

A semilattice is said to be *linear* if its partial order (defined as usual by $x \leq y \iff x = xy$) is a chain. The variety generated by complex algebras of linear semilattices is denoted by LBSL.

Theorem 1. [1] *LBSL is the variety of Boolean algebras with a commutative associative idempotent operator, i.e., square-increasing is strengthened to $x = x^2$.*

The question whether RBSL has a finite equational basis is currently still open. The following additional (quasi)identities are known.

Lemma 1. *Algebras in RBSL also satisfy the following axioms:*

1. $x \wedge 1y \leq xy$ (*we assume $\cdot$ has priority over $\vee, \wedge$*).
2. $x(xy - x) \leq x^2 \vee (xy - x)^2$ (*from* [1])

3. $u \le yz \implies xu \le (xz \wedge v)y \vee (xz - v)u$
4. $xy \le x \vee y \implies x^2 \wedge y^2 \le xy$
5. $x \le yz \le x \vee y, x \vee w \le yw, x \wedge w = 0, xw \le x$ *and* $zw \le w \implies x \le y^3$

There are finite BAOs that show these formulas are not implied by the BSL identities.

Let B be a Boolean semilattice that satisfies the identity $x \wedge 1y \le xy$ and define the relation $x \sqsubseteq y$ if and only if $x \le xy$. This relation is reflexive since $x \le x^2$, and transitivity follows from the identity, hence $\sqsubseteq$ is a preorder. If B is atomic (in particular, if it is finite) we restrict $\sqsubseteq$ to the atoms $A = \mathrm{At}(B)$.

Lemma 2. *If A is embedded in the complex algebra of a finite semilattice then $\sqsubseteq$ is a partial order.*

A semilattice is called a *tree-semilattice* if its partial order is a tree (i.e., it has a least element and every principal downset is a chain). A BSL is tree-representable if it is embedded in the complex algebra of a tree-semilattice. The variety generated by tree-representable Boolean semilattices is denoted by TBSL. It properly contains LBSL and is a proper subvariety of RBSL.

Theorem 2. *Let A be a finite Boolean semilattice that satisfies $x \wedge 1y \le xy$ and assume $\sqsubseteq$ is a partial order on the atoms of A. Then the following are equivalent.*

- *A is tree-representable,*
- *A is embedded in the complex algebra of a finite tree-semilattice,*
- *A satisfies the identity $(x \wedge xy)z \wedge -x \le yz$.*

It is conjectured that the variety TBSL is axiomatized relative to BSL by the identity $(x \wedge xy)z \wedge -x \le yz$.

A Boolean semilattice is called *integral* if $xy = 0$ implies $x = 0$ or $y = 0$. This property holds in all complex algebras of semilattices and hence also in all subalgebras of them. It is easy to see that all BSLs that satisfy $x \wedge 1y \le xy$ and have up to 4 elements are in RBSL. A computer calculation shows that there are (up to isomorphism) 79 integral Boolean semilattices with 8 elements that satisfy the additional quasiequations in Lemma 1. Of these, 13 have $\sqsubseteq$ as a partial order on the atoms and they are all embedded in the complex algebra of some finite semilattice, hence in RBSL. There are 25 that satisfy the identity $(x \wedge xy)z \wedge -x \le yz$ (including 9 from the previous 13) and they are also in RBSL. Many of the remaining 50 algebras are known to be in RBSL as well, but currently this is not known for all of them.

References

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