Representation of Partial Traces

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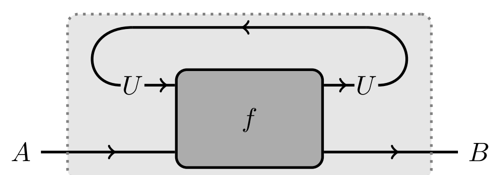
The notion of trace in a monoidal category has been introduced by Joyal Street and Verity [4] to give a categorical account of a situation occurring in very different settings: linear algebra, knot theory, proof theory... The basic idea is that a trace is an operation that to any morphism

$$f : A \otimes U \to B \otimes U$$

in a monoidal category associates a new morphism

$$\mathbf{Tr}^U(f) : A \to B$$

with the intuition that this operation can be understood as a *feedback over U*, which is acknowledged in the graphical language for these categories by depicting $\mathbf{Tr}^U(f)$ as



This operation has to satisfy a number of axioms that capture formally what is expected of a notion of feedback in a monoidal category.

More recently Haghverdi and Scott [2] introduced the notion of *partial trace*, accounting for the fact that the trace operation can be only partially defined. This is a situation that occur very naturally in practice: think of the trace in an infinite-dimensional Hilbert space or feedback loops in circuits, for instance. This notion was further studied by Malherbe, Scott and Selinger [5].

Their main result was a representation theorem for partial traces: any partially traced category can be embedded in a fully traced category, via an embedding preserving the traced structure. They also obtained an universal property for their construction. This result allow to treat partial traces as total ones (in particular carrying computations in the graphical language) without having to worry at each step whether everything is defined.

We propose in this talk to revisit this theorem and provide a simpler proof. Indeed the original approach to this involved Freyd's paracategories [3] and a partially defined version of the $\mathbf{Int}(\cdot)$ [4] construction of totally traced categories, leading to a relatively complex proof and delicate argumentation dealing with partially defined operations.

Sketch of the proof

We start with a partially traced category $\mathbf{C}$ and apply to it a construction — used in particular in categorical studies of automata theory [1] — $\mathcal{D}(\cdot)$ that endow morphisms with a "private" part that can be thought of as a state space.

We further quotient $\mathcal{D}(\mathbf{C})$ to enforce some needed equations and go on proving that this quotiented category enjoys a total trace: intuitively, tracing becomes moving an interface to the private part. Moreover, $\mathbf{C}$ embeds in the quotient of $\mathcal{D}(\mathbf{C})$ via a trace-preserving embedding.

Then, using the compatibility of the equivalence relation used for the quotient and trace-preserving functors we are able to prove the universality of our construction.

References

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