Injectivity of relational semantics for (connected) MELL proof-structures via Taylor expansion

Giulio Guerrieri[1], Luc Pellissier[2], and Lorenzo Tortora de Falco[3]

[1] PPS, UMR 7126, Univ Paris Diderot, Sorbonne Paris Cit, F-75205 Paris, France
[2] LIPN, UMR 7030, Universit Paris 13, Sorbonne Paris Cit, Villetaneuse, France
[3] Dipartimento di Matematica e Fisica, Universit Roma Tre, Roma, Italy

Given a syntax $\mathcal{S}$ endowed with some rewrite rules, and given a denotational semantics $\mathcal{D}$ for $\mathcal{S}$ (i.e. a semantics which gives to any term t of $\mathcal{S}$ an interpretation $[\![t]\!]_{\mathcal{D}}$ that is invariant under the rewrite rules), we say that $\mathcal{D}$ is *injective* with respect to $\mathcal{S}$ if, for any two normal terms t and t' of $\mathcal{S}$, $[\![t]\!]_{\mathcal{D}} = [\![t']\!]_{\mathcal{D}}$ implies $t = t'$. In categorical terms, injectivity corresponds to faithfulness of the "interpretation-functor" from $\mathcal{S}$ to $\mathcal{D}$; it is a natural and well studied question for denotational semantics of λ-calculi and term rewriting systems (see [7,10]). All (positive) results of injectivity of denotational models with respect to some syntax fit in the general perspective of abolishing the traditional distinction between syntax and semantics.

Starting from investigations on denotational semantics of System F (second order typed λ-calculus), in 1987 Girard [8] introduced linear logic (LL), a refinement of intuitionistic logic. He defines two new modalities, ! and ?, giving a logical status to structural rules and allowing one to distinguish between linear resources (i.e. usable exactly once during the cut-elimination process) and resources available at will (i.e. erasable and duplicable during the cut-elimination process). One of the main features of LL is the possibility of representing proofs geometrically (so as the λ-calculus terms) by means of particular graphs called proof-structures. Among proof-structures it is possible to characterize "in a geometric way" the ones corresponding to proofs in LL sequent calculus through the Danos-Regnier correctness criterion [2] (see also [11]): roughly speaking, a proof-structure is a proof-net (i.e. it corresponds to a proof in LL sequent calculus) if and only if it fulfils some conditions about acyclicity and connectedness (ACC).[4]

Ehrhard [3] introduced finiteness spaces, a denotational model of LL (and λ-calculus) which interprets formulas by topological vector spaces and proofs by analytical functions: in this model the operations of differentiation and the Taylor expansion make sense. Ehrhard and Regnier [4,5,6] internalized these operations in the syntax and thus introduced differential linear logic DiLL_0 (which

[4] Strictly speaking, this equivalence holds only in some fragments of LL, for example the multiplicative one (MLL) without $\bot$. In larger fragments of LL, as for instance MELL (the multiplicative-exponential fragment of LL, sufficiently expressive to encode the λ-calculus) one only has that all proof-nets are ACC proof-structures, but to obtain the converse ACC is not sufficient, additional hypotheses are required.

encodes the resource λ-calculus, see [5]), where the promotion rule (the only one in LL which is responsible for introducing the !-modality and hence for creating resources available at will, marked by boxes in LL proof-structures) is replaced by three new "finitary" rules introducing !-modality which are perfectly symmetric to the rules for the ?-modality: this allows a more subtle analysis of the resources consumption during the cut-elimination process. At the syntactic level, the Taylor expansion decomposes a LL proof-structure in a (generally infinite) formal sum of DiLL_0 proof-structures, each of which contains resources usable only a fixed number of times. Roughly speaking, each element of the Taylor expansion $\mathcal{T}(\pi)$ of a LL proof-structure π is a DiLL_0 proof-structure obtained from π by replacing each box B in π with n_B copies of its content (for any $n_B \in \mathbb{N}$), recursively.

The question of injectivity of some set-based denotational model (i.e. denotational model whose interpretations are sets) with respect to LL proof-structures has been first addressed in [11] and some remarkable (positive and negative) results are in [11,9,1]: in particular, in [11] Tortora de Falco has shown that coherent semantics in not injective with respect to MELL proof-nets, but it is injective with respect to some fragments of MELL, for instance the fragment of MELL that encodes the λ-calculus.

Our contribution aims at looking further into the relationship between the Taylor expansion and the relational model. The relational model is one of the well-known and simplest denotational semantics of LL and λ-calculus: it interprets LL proof-structures as morphisms in the category of sets and relations. Our work proves that the relational semantics is injective with respect to MELL proof-structures fulfilling some condition about connectedness. The injectivity of the relational model in the similar case of MELL proof-structures without weakenings has already recently been proved by de Carvalho and Tortora de Falco in [1]. Our proof follows a different and more geometrical approach based on the notion of Taylor expansion; it represents both a simplification and a generalization of the result contained in [1]:

1. We notice that, given a cut-free and η-expanded (i.e. with atomic axioms) MELL proof-structure π, each point of the Taylor expansion of π is isomorphic to one and only one element of the set of injective points of the interpretation of π in the relational model, quotiented by the equivalence relation induced by atoms renaming. (This does not hold if π contains cuts, consistently with the idea that the Taylor expansion of a MELL proof-structure can be seen as an object between syntax and semantics). We can thus use a graph-theoretic representation of the elements of the relational interpretation of a cut-free and η-expanded MELL proof-structure by means of DiLL_0 proof-structures.
2. We show that every box-connected[5] MELL proof-structure π is uniquely determined by the point of order 2 in its Taylor expansion (i.e. the DiLL_0

[5] Informally, a MELL proof-structure is *box-connected* if, for every box B, all the content of B is "accessible" from its !-door. Our notion of accessibility is related to that of empire, a well-known tool of the theory of MLL proof-nets introduced by Girard in [8]. Notice that a box-connected MELL proof-structure might contain

proof-structure obtained from π by taking exactly two copies of the content of each box of π, recursively)[6]: if π_1 and π_2 are two box-connected MELL proof-structures (possibly with cuts) such that their respective Taylor expansions $\mathcal{T}(\pi_1)$ and $\mathcal{T}(\pi_2)$ have the same point of order 2, then $\pi_1 = \pi_2$. In order to do a comparison with mathematical analysis, the analogous of this result is that analytical functions fulfilling some condition are uniquely determined by their second derivative!

3. As a corollary of points (1) and (2), we show that the relational model is injective with respect to box-connected MELL proof-structures: given two cut-free and η-expanded box-connected MELL proof-structures, if they have the same interpretation in relational semantics then they are identical.

We would like to stress that the box-connectedness hypothesis in our results of points (2) and (3) is quite general and not *ad hoc*: all ACC MELL proof-structures, all MELL proof-nets without $\perp$ and weakening, and all MELL proof-structures that are the translation of λ-terms are box-connected. Moreover, box-connectedness is preserved under cut-elimination.

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cuts, while in [1] MELL proof-structures are all cut-free: in this respect our result generalizes [1].

[6] Following the approach of [1], the order of the point of the Taylor expansion allowing one to distinguish two different MELL proof-structures π and π' depends on π and π': in this respect our result simplifies [1].