Yet another ring-theoretic characterization of P-frames

Oghenetega Ighedo

University of South Africa, Department of Mathematical Sciences,
Pretoria, South Africa
http://www.unisa.ac.za

Abstract. Call a ring *z-good* if it has the property that an ideal in it is a z-ideal if and only if its radical is a z-ideal. By first showing that a z-good ring is von Neumann regular if and only if every prime ideal in it is a z-ideal, I will characterize P-frames as precisely those L for which every prime ideal in the ring $\mathcal{R}L$ (the ring of continuous real-valued functions on L) is a z-ideal. Furthermore, I will show that this characterization still holds if prime ideals are replaced by essential ideals, radical ideals, convex ideals, or absolutely convex ideals.

Keywords: P-frame; ring of real-valued continuous functions on a frame; d-ideal; z-ideal

1 Introduction

Throughout the talk, the term "ring" will mean a commutative ring with identity. In [5], Mason defines an ideal I of a ring A to be a *z-ideal* if whenever a and b in A belong to the same maximal ideals of A and a is in I, then b too is in I. Maximal ideals, minimal prime ideals, and intersections of z-ideals are z-ideals. He then characterises, among reduced rings (i.e. rings with no nonzero nilpotent elements), von Neumann regular rings as exactly those in which every ideal is a z-ideal [5, Theorem 1.2]. Recall that A is *von Neumann regular* if for every $a \in A$ there exists $b \in A$ such that $a = a^2b$.

A Tychonoff space X is called a *P-space* in case every zero-set of X is open. These spaces are precisely those X for which the ring $C(X)$ is von Neumann regular. In pointfree topology, P-frames are defined by a frame-theoretic enunciation of the zero-set condition defining P-spaces. Namely; L is a *P-frame* if every cozero element of L is complemented. As Banaschewski and Hong show in [3], L is P-frame if and only if the ring $\mathcal{R}L$ of real-valued continuous functions on L is von Neumann regular.

Numerous characterisations of P-spaces have their localic counterparts (see [1] and [4]). In [6] it is stated as an exercise (Problem 14B.4) that X is a P-space precisely when every prime ideal of $C(X)$ is a z-ideal. There are two suggested hints; one using points of the space X (which wouldn't work for frames as tere are P-frames with no points), and the other using upper ideals.

In this talk I will outline a proof that L is a P-frames if and only if every prime ideal of $\mathcal{R}L$ is a z-ideal which is completely different from those suggested for P-spaces in [6].

2 Some ring-theoretic terms to be used

We have already recalled what a z-ideal. An ideal of a ring is *essential* if it intersects every nonzero ideal of the ring nontrivially. Recall that a ring A is *reduced* if it has no nonzero nilpotent elements. For a reduced ring (like all rings $\mathcal{R}L$, and hence $C(X)$) an ideal I is essential precisely when 0 is the only element of the ring that annihilates every element of I.

The *radical* of an ideal I of a ring A is the ideal

$$\sqrt{I} = \{a \in A \mid a^n \in I \text{ for some } n \in \mathbb{N}\}.$$

A *radical ideal* is an ideal which equals its radical. Let A be a lattice-ordered ring (usually abbreviated as "ℓ-ring"). An ideal $I \subseteq A$ is said to be *convex* if, for any $a, b \in A$,

$$0 \leq a \leq b \text{ and } b \in I \implies a \in I.$$

On the other hand, I is called *absolutely convex* if if, for any $a, b \in A$,

$$|a| \leq |b| \text{ and } b \in I \implies a \in I.$$

3 Main results

The proofs of the main results rely on the following proposition culled from [2]. By a *function ring* we mean a ring of the form $\mathcal{R}L$, and by a *classical function ring* we mean one of the form $C(X)$.

Proposition 1. *The following are equivalent for any archimedean f-ring A with unit:*

1. *A is isomorphic to some a functional ring.*
2. *A is a homomorphic image of some classical function ring.*
3. *A is isomorphic to the ω-updirected union of sub-ℓ-rings isomorphic to classical function rings.*

The idea of the proof is first to show that function rings are z-good (see the abstract for the definition). Some of the needed lemmas are the following.

Lemma 1. *A z-good ring is von Neumann regular iff every prime ideal in it is a z-ideal.*

Lemma 2. *$\mathcal{R}L$ is a z-good ring.*

These two are enough to prove the following result.

Proposition 2. *The following are equivalent for a completely regular frame L.*

1. *L is a P-frame.*
2. *Every essential ideal in $\mathcal{R}L$ is a z-ideal.*
3. *Every radical ideal in $\mathcal{R}L$ is a z-ideal.*

For the next characterisations we need the following lemma.

Lemma 3. *Every radical ideal in $\mathcal{R}L$ is absolutely convex.*

Armed with this lemma, we arrive at the following characterisations.

Proposition 3. *The following are equivalent for a completely regular frame L.*

1. *L is a P-frame.*
2. *Every convex ideal in $\mathcal{R}L$ is a z-ideal.*
3. *Every absolutely convex ideal in $\mathcal{R}L$ is a z-ideal.*

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