On pretransitive logics of finite depth

Andrey Kudinov*† and Ilya Shapirovsky*

*Institute for Information Transmission Problems of the Russian Academy of Sciences
†Higher School of Economics
†Moscow Institute of Physics and Technology
{kudinov, shapir}@iitp.ru

We study logics with expressible "transitive closure" modality (*pretransitive* logics). In such logics we can express formulas of finite depth. We prove the finite model property for a family of pretransitive logics of finite depth.

There is an old problem about decidability of logics $\mathsf{K}^{\mathsf{m}}_{\mathsf{n}} = \mathsf{K} + \Box^m p \to \Box^n p$, where $\Box^m$ is the sequence of m boxes. For the case when $m \leq 1$ or $n \leq 1$, and for the trivial case $m = n$ the finite model property (FMP) is known. As for the other cases it is unknown whether $\mathsf{K}^{\mathsf{m}}_{\mathsf{n}}$ has FMP or even if it is decidable.

If $n > m$ then the logic $\mathsf{K}^{\mathsf{m}}_{\mathsf{n}}$ is *pretransitive*[1], which means that we can express the truth in a point-generated submodel. Formally, L is *pretransitive* if there exists a formula $\chi(p)$ with a single variable p such that for any Kripke model M with $\mathsf{M} \vDash \mathsf{L}$ and for any w in M we have

$$\mathsf{M}, w \vDash \chi(p) \Leftrightarrow \forall u (w R^* u \Rightarrow \mathsf{M}, u \vDash p),$$

where R^* is the transitive reflexive closure of the accessibility relation of M. It is known [2] that L is pretransitive iff for some $k \geq 0$ it contains the formula of *k-transitivity* $\Box^{\leq k} p \to \Box^{k+1} p$, where

$$\Box^{\leq k} \varphi = \bigwedge_{i=0}^{k} \Box^i \varphi.$$

We put $\Box^* \varphi = \Box^{\leq k} \varphi$ for the least such k. In particular, for $\mathsf{K}^{\mathsf{m}}_{\mathsf{n}}$ $(n > m)$ $\Box^* \varphi = \Box^{\leq n-1} \varphi$.

For logics $\mathsf{K}_{\leq m} = \mathsf{K} + \Box^{\leq m} p \to \Box^{m+1} p$, FMP is also unknown for all $m > 1$ (the logic $\mathsf{K}_{\leq 1} = \mathsf{wK4}$ is known to have FMP).

Since logics $\mathsf{K}^{\mathsf{m}}_{\mathsf{n}}$ and $\mathsf{K}_{\leq \mathsf{m}}$ are Kripke complete, to prove FMP we can proceed in the following standard way: for an L-frame F and a satisfiable in F formula φ to construct a finite L-frame in which formula φ is still satisfiable. We do not know how to do this for arbitrary $\mathsf{K}^{\mathsf{m}}_{\mathsf{n}}$- and $\mathsf{K}_{\leq \mathsf{m}}$-frames, but we find a way to construct filtrations for all such pretransitive frames of finite depth. Here *the depth of a frame* (W, R) is the maximal length of chains in the skeleton of (W, R^*).

Lemma 1. *Let* L *be one of the logics* $\mathsf{K}^{\mathsf{m}}_{\mathsf{n}}$, $\mathsf{K}_{\leq \mathsf{m}}$, $n > m \geq 1$. *If a formula is satisfiable in an* L*-frame of finite depth* h*, then it is satisfiable in a finite* L*-frame*

[1] or *conically expressive* [2], or *weakly transitive* [5], or $(n-1)$*-transitive* [1]

of depth h; the size of the latter frame is bounded by

$$2^{2^{\cdot^{\cdot^{\cdot^{2^l}}}}}\Big\}h,$$

where l is the length of the formula.

As well as for the logic S4, for a pretransitive logic we can define *formulas of finite depth*:

$$B_1 = p_1 \to \Box^*\Diamond^* p_1, \quad B_{h+1} = p_{h+1} \to \Box^*(\Diamond^* p_{h+1} \vee B_h).$$

Proposition 1. *For a pretransitive logic* L *and an* L*-frame* F, $F \vDash B_h$ *iff the depth of* F *is no greater then* h.

The extension of a pretransitive logic L with B_h is denoted by $\mathsf{L}.B_h$. The following theorem is an analogue of known facts about such extensions of S4.

Theorem 1. *Let* L *be a pretransitive logic. Then*

1. $\mathsf{L}.B_1 \supseteq \mathsf{L}.B_2 \supseteq \mathsf{L}.B_3 \supseteq \ldots \supseteq \mathsf{L}$.
2. *If* L *is consistent then* $\mathsf{L}.B_1$ *(and, consequently, each* $\mathsf{L}.B_h$*) is consistent.*
3. *If* L *is canonical then each* $\mathsf{L}.B_h$ *is canonical.*

Corollary 1. *For all* $n > m \geq 1$, $h \geq 1$, *logics* $\mathsf{K}_n^m.B_h$ *and* $\mathsf{K}_{\leq m}.B_h$ *are canonical and, hence, Kripke complete.*

In general, pretransitive logics of finite depth are much more complicated then their analogues above S4. For example, all logics $\mathsf{S4}.B_h$ are locally tabular, $\mathsf{S4}.B_1 = \mathsf{S5}$ is pretabular, while no local tabularity or pretabularity holds even for the "simplest" nontransitive pretransitive logic $\mathsf{K}_{\leq 2}.B_1$ [7,4]. The same true for all logics $\mathsf{K}_n^m.B_h$, $\mathsf{K}_{\leq m}.B_h$ $(n > m \geq 2)$, since they are smaller then $\mathsf{K}_{\leq 2}.B_1$.

However, all these logics have FMP. Even in the case of depth 1 (the case when we consider pretransitive analogues of S5, i.e., when R^* is an equivalence relation) the proof is not trivial (especially for logics $\mathsf{K}_n^m.B_1$ with $n > m+1$). FMP for logics $\mathsf{K}_{\leq \mathsf{m}} + p \to \Box^{\leq m}\Diamond^{\leq m} p = \mathsf{K}_{\leq \mathsf{m}}.B_1$ was proved in [3], and for pretransitive logics $\mathsf{K}_\mathsf{n}^\mathsf{m}.B_1$ — in [6].

Our main result is the following.

Theorem 2. *For all* $n > m \geq 1$, $h \geq 1$, $\mathsf{K}_n^m.B_h$ *and* $\mathsf{K}_{\leq m}.B_h$ *have FMP.*

Corollary 2. *Let* L *be one of the logics* $\mathsf{K}_\mathsf{n}^\mathsf{m}$, $\mathsf{K}_{\leq \mathsf{m}}$, $n > m \geq 1$.

$$\mathsf{L} \textit{ has FMP iff } \mathsf{L} = \bigcap_{h \geq 1} \mathsf{L}.B_h.$$

References

1. Chagrov, A., Zakharyaschev, M.: Modal Logic, Oxford Logic Guides, vol. 35. Oxford University Press (1997)
2. Gabbay, D., Shehtman, V., Skvortsov, D.: Quantification in Nonclassical Logic. Elsevier (2009)
3. Jansana, R.: Some logics related to von Wright's logic of place. Notre Dame J. Formal Logic 35(1), 88–98 (01 1994)
4. Kostrzycka, Z.: On non-compact logics in NEXT(KTB). Mathematical Logic Quarterly 54(6), 617–624 (2008)
5. Kracht, M.: Tools and Techniques in Modal Logic. Elsevier (1999)
6. Kudinov, A., Shapirovsky, I.: Finite model property of pretransitive analogs of S5. In: Topology, Algebra and Categories in Logic (TACL 2011). pp. 261–264 (2011)
7. Miyazaki, Y.: Normal modal logics containing KTB with some finiteness conditions. AiML-2004: Advances in Modal Logic p. 264 (2004)