Relational semantics via TiRS graphs

Willem Conradie and Andrew Craig

Department of Pure and Applied Mathematics, University of Johannesburg
wconradie@uj.ac.za and acraig@uj.ac.za

The availability of simple and intuitive relational semantics has made an important contribution to the study of many non-classical logics. This has particularly been the case for such logics with a classical or intuitionistic propositional base, for example modal logic and intuitionistic modal logic. However, it has been more difficult to use relational semantics to study logics whose algebraic semantics are given by lattices that are not necessarily distributive. This is a result of the fact that the duality for algebras whose underlying lattices are non-distributive is far more complicated than the existing dualities for distributive lattices and Boolean algebras. This difficulty has resulted in relational semantics that do not have the same nice intuition that has aided the study of logics based on Boolean algebras or distributive lattices.

Some of the existing approaches have made use of Urquhart's representation [10]; see for example [7]. The RS frame semantics described by Gehrke [8] are an attempt to by-pass the problematic representation theorems and rather access the relational structures via the canonical extension. An RS frame is a two-sorted relational structure of the form $\mathbb{F} = (X, Y, R)$ where X is a set of worlds, Y a set of co-worlds, and R a relation from X to Y. These structures have been applied to linear logics [3], logics with negation [1], as well as the Lambek-Grishin calculus [2].

We provide a general relational semantics for logics, the algebraic semantics of which are based on bounded lattices with additional operators. This semantics takes the form of TiRS graphs (a TiRS graph is a set X equipped with a binary relation E satisfying certain conditions) together with relations used to interpret the other connectives of the language. In this way our semantics is similar to classical Kripke frames (where E is the diagonal relation) and intuitionistic Kripke frames (where E is a partial order) and indeed has these as special cases.

The idea of working with a graph (X, E) has its roots in the dual representation theorem of Ploščica [9]. Recent work in [4,5] has shown how to construct the canonical extension of a lattice from Ploščica's representation. The untopologised graphs that occur in Ploščica's representation were characterised in [6] where they are called *TiRS graphs*.

Urquhart's dual representation for arbitrary bounded lattices [10] uses a particular set of disjoint filter-ideal pairs as the underlying set of the dual space. These pairs are "maximal" in the sense that neither the filter nor the ideal can be extended without there being an element of the lattice in their intersection. This set of filter-ideal pairs is then equipped with two quasi-orders.

Ploščica provided a variation on this representation by viewing the underlying set as a set of maximal partial homomorphisms. Instead of maximal disjoint filter-ideal pairs, the underlying set of the dual representation for a bounded lattice $\mathbf{L}$ is the set of maximal partial homomorphisms from $\mathbf{L}$ into $\underline{\mathbf{2}}$ (the two-element bounded lattice). Such a partial homomorphism is maximal in the sense that its domain cannot be extended without it failing to be a lattice homomorphism. The set is denoted by $\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$. (It is easily shown that the set $\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$ is in a one-to-one correspondence with Urquhart's maximal filter-ideal pairs.) The major change in Ploščica's representation was to replace the two quasi-orders with a binary relation E on the set $\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$.

For $f, g \in \mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$, we have

$$fEg \quad \Leftrightarrow \quad f^{-1}(1) \cap g^{-1}(0).$$

Although we will consider abstract TiRS graphs as the setting for our relational semantics, it is useful to remember the origin of these structures as the untopologised dual spaces of bounded lattices. When f and g are thought of as worlds in our relational structures, the above definition can be read as follows: "there is no proposition asserted at f and denied at g". We further interpret the E relation by thinking of it in the following way

$$fEg \quad \Leftrightarrow \quad \text{“} f \text{ trusts } g\text{”}.$$

When a relational structure (doubly-ordered set/RS frame/TiRS graph) is used to model a lattice-based algebra with an additional n-ary operator, an extra $(n+1)$-ary relation is added to the relational structure. Some compatibility is then required between the extra relation and the Galois-closed subsets of the relational structure. This compatibility can often be complicated to describe, but in our setting we have a relatively simple description of the conditions. As an example, consider a TiRS graph $\mathbf{X} = (X, E)$ equipped with an additional binary relation, $R_\diamond$. We require the following compatibility condition between $R_\diamond$ and E:

$$\forall x \forall w \Big(\forall y (xEy \Rightarrow \exists z (zEy \wedge zR_\diamond w)) \Rightarrow xR_\diamond w \Big) \tag{1}$$

Condition (1) can equivalently be written as

$$\forall x \forall w (xR_\diamond w \vee \exists y (xEy \wedge \forall z (zEy \Rightarrow \neg zR_\diamond w))) \tag{2}$$

If we do not require condition (1), for a modal TiRS frame $\mathbf{X}_\diamond = (X, E, R_\diamond)$ our relational semantics for the *assertion* of the $\diamond$ operator is defined as follows:

$$x \Vdash \diamond\varphi \qquad \text{iff} \qquad \forall y (xEy \Rightarrow \exists z (yR_\diamond z \text{ and } z \Vdash \varphi))$$

The propositional formula $\diamond\psi$ is *denied* at a world x if the following condition is satisfied:

$$x \succ \diamond\psi \qquad \text{iff} \qquad \forall y (xR_\diamond y \Rightarrow \neg (y \Vdash \psi))$$

At a world x, certain propositional formulas might neither be asserted nor denied. This is a result of the world x being (maximally) *partial*, which in turn is a result of the lack of distributivity.

For a formula φ, let $[\![\varphi]\!]$ be the set of worlds at which φ is asserted. If we do require condition (1) on $\mathbf{X}_\diamond = (X, E, R_\diamond)$, we then have the conditions:

$$x \Vdash \diamond\varphi \quad \text{iff} \quad x \in R_\diamond^{-1}([\![\varphi]\!]) \qquad \text{and} \qquad x \succ \diamond\psi \quad \text{iff} \quad \forall y (xR_\diamond y \Rightarrow \neg (y \Vdash \psi))$$

We apply our new setting to other modal operators, as well as to the connectives of substructural logics, and obtain intuitive relational semantics for these settings. Completeness results for certain examples can be obtained from the correspondence between TiRS graphs and TiRS frames shown in [6].

References

1. Almeida, A.: Canonical extensions and relational representations of lattices with negation, Studia Logica **91**, 171–199 (2009)
2. Chernilovskaya, A., Gehrke, M., van Rooijen, L.: Generalized Kripke semantics for the Lambek-Grishin calculus, Log. J. IGPL **20**, 1110–1132 (2012)
3. Coumans, D., Gehrke, M., van Rooijen, L.: Relational semantics for full linear logic, Journal of Pure and Applied Logic **12**, 50–66 (2014)
4. Craig, A.P.K., Haviar, M., Priestley, H.A.: A fresh perspective on canonical extensions for bounded lattices, Appl. Categ. Structures **20**, 725–749 (2013)
5. Craig, A.P.K., Haviar, M.: Reconciliation of approaches to the construction of canonical extensions of bounded lattices, Math. Slovaca **64**, 1335–1356 (2014)
6. Craig, A.P.K., Haviar, M., Gouveia, M.J.: TiRS graphs and TiRS frames: a new setting for duals of canonical extensions, Algebra Universalis, to appear.
7. Dzik, W., Orlowska, E., van Alten, C.: Relational representation theorems for lattices with negations: a survey, In: TARSKI II, de Swart et al. (Eds), LNAI 4342, 245–266 (2006)
8. Gehrke, M.: Generalized Kripke frames, Studia Logica **84**, 241–275 (2006)
9. Ploščica, M.: A natural representation of bounded lattices, Tatra Mountains Math. Publ. **5**, 75–88 (1995)
10. Urquhart, A.: A topological representation theory for lattices, Algebra Universalis **8**, 45–58 (1978)