Relational lattices via duality

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Relational lattices were introduced by Spight and Tropashko in a series of preprints [6,7,8]; equational principles were shown to hold in a signature strictly larger than the one of lattice theory. Litak, Mikulás and Hidders [2] exhibited later two non-trivial lattice identities holding in this class.

Relational lattices are interesting in view of their close connection with database theory. An element of the relational lattice $\mathsf{R}(D, A)$ is a table whose entries are from D and whose header is a subset of A, a total set of attributes; mathematically, a table is a pair (X, T) with $X \subseteq A$ and $T \subseteq D^X$. Given two tables (X_i, T_i), $i \in 2$, the order is given by $(X_0, T_0) \leq (X_1, T_1)$ iff $X_1 \subseteq X_0$ and $T_0 \subseteq \{f \mid f_{\restriction X_1} \in T_1\}$. This ordered set is a lattice whose operations coincide with two well-known operations in the theory and practice of databases. The meet is the *outer-join* of tables:

$$(X_0, T_0) \wedge (X_1, T_1) = (X_0 \cup X_1, T), \quad T = \{f \in D^{X_0 \cup X_1} \mid f_{\restriction X_i} \in T_i, i = 0, 1\},$$

while the join is the *inner-union* of tables:

$$\begin{aligned}(X_0, T_0) \vee (X_1, T_1) = (X_0 \cap X_1, T)\,,& \\ T = \{f \in D^{X_0 \cap X_1} \mid \exists i \in 2, g \in T_i, f = g_{\restriction X_0 \cap X_1}\}.&\end{aligned}$$

It was argued in [2] that $\mathsf{R}(D, A)$ arises via a closure operator on the powerset $P(A + D^A)$ and, at the same time, as the category of elements of the functor $F_{A,D}$, from $P(A)^{op}$ to $\mathcal{JSL}_{\mathsf{c}}$ (the category of complete join-semilattices), sending $X \subseteq A$ to D^X and then D^X covariantly to $P(D^X)$.

We depend on these observations in our quest for further knowledge on the equational theory of relational lattices. We tackle this problem using the duality developed in [3] and further understood through our collaboration with F. Wehrung [4,5]. For a complete lattice L, a join-cover of $x \in L$ is a subset $Y \subseteq L$ such that $x \leq \bigvee Y$. Relational lattices are (complete) spatial lattices such that every join-cover of a completely join-irreducible refines to a minimal one. This property allows to define the dual structure, known as the OD-graph, of any spatial lattice. This is the structure $\langle \mathcal{J}(L), \leq, \lhd \rangle$ with $\mathcal{J}(L)$ the set of completely join-irreducible elements, $\leq$ the restriction of the order to $\mathcal{J}(L)$, and the relation $j \lhd C$ holds when $j \in \mathcal{J}(L)$, $C \subseteq \mathcal{J}(L)$, and C is a minimal join-cover of j.

We identify completely join-irreducible elements of $\mathsf{R}(D, A)$ with elements of the disjoint sum $A + D^A$, see [2]. Being atomistic, the order on completely join-irreducible elements is discrete. All the elements of A are join-prime; whence the only minimal join-cover of $x \in A$ is the trivial one, $\{x\}$. The minimal join-covers of elements in D^A can be easily described via an ultrametric distance valued in the join-semilattice $P(A)$; this is, morally, the Hamming's one, $\delta(f, g) = \{x \in$

$A \mid f(x) \neq g(x)\}$. Whenever $f, g \in D^A$ we have $f \lhd \delta(f,g) \cup \{g\}$ and these are all the minimal join-covers of f.

We give proofs, relying on this duality, that identities AxRL1 and AxRL2 from [2] hold in relational lattices. More importantly, we exhibit an infinite collection of identities (including but not restricted to the ones given below) that holds in the class of relational lattices. Some properties of the OD-graph, of the ultrametric space, and of the functor $F_{A,D}$, play a key role when showing these identities valid. Namely:

P1. Every non-trivial minimal join-cover contains at most one join-irreducible element which is not join-prime.

P2. The distance function (whence the coverings) is symmetric.

P3. The functor $F_{A,D} : P(A)^{op} \longrightarrow \mathcal{JSL}_{\mathsf{c}}$ sends a pullback square (i.e., a square of inclusions with objects $X \cap Y, X, Y, Z$) to a square satisfying the Beck-Chevalley condition. This property include a form of Malchev condition, stating that a collection of congruences commute.

We show that property P1 of an OD-graph is definable by the lattice identity

$$(x \wedge (y_l \vee z_l \vee w) \vee (x \wedge (y_r \vee z_r \vee w)) = (x \wedge (y_r \vee z_l \vee w)) \vee (x \wedge (y_l \vee z_r \vee w))$$

with $y_l := y_0 \wedge (y_1 \vee y_2)$, $y_r := (y_0 \wedge y_1) \vee (y_0 \wedge y_2)$, and similarly for z_l and z_r. Noticing that $y_r \leq y_l$ and $z_l \leq z_r$, the reader will recognize the similarity with AxRL2. This identity is strictly stronger than AxRL2; it is possible to derive from it the similar identities with y_l, y_r, z_l, z_r instantiated by any tuple of lattice terms such that the equalities $y_l = y_r$ and $z_l = z_r$ hold on distributive lattices. In relational lattices, every minimal join-cover contains exactly one join-irreducible which is not join-prime; yet this variant of P1 turns out not to be definable by lattice identities.

It is possible to further investigate properties P2 and P3 in a larger setting, using semidirect products of join-semilattices (the Grothendieck construction) and the particular case when such a semidirect product arises from a (generalized, possibly non-symmetric) ultrametric space valued on a powerset—see [1] for usual ultrametric spaces and lattices. In presence of P1, symmetry and the Beck-Chevalley property can be understood as properties of the OD-graph. Symmetry is the following property: if $k_0 \lhd C \cup \{k_1\}$ with k_1 not join-prime, then $k_1 \leq \bigvee C \vee k_0$—i.e. $k_1 \lhd D$ for some D which refines $C \cup \{k_1\}$. The Beck-Chevalley property is the following one: if $k_0 \lhd C_0 \cup C_1 \cup \{k_2\}$ with k_2 not join-prime and $C_0, C_1, \{k_2\}$ pairwise disjoint, then $k_0 \lhd C_0 \cup \{k_1\}$ and $k_1 \lhd C_1 \cup \{k_2\}$ for some completely join-irreducible element k_1. Assuming P1, the following is a group of identities—valid on relational lattices—by which we can characterize lattices whose OD-graph are symmetric and have the Beck-Chevalley property:

$$\begin{aligned} x \wedge (y \vee z) &= \\ &\quad (x \wedge (y \vee (z \wedge (x \vee y)))) \vee (x \wedge (z \vee (y \wedge (x \vee z)))) \,, \\ x \wedge ((y \wedge z) \vee (y \wedge x) \vee (z \wedge x)) &= (x \wedge y) \vee (x \wedge z) \,, \\ x \wedge ((x \wedge y) \vee z_l) &= (x \wedge ((x \wedge y) \vee z_r)) \vee (x \wedge z_l) \,. \end{aligned}$$

References

1. Lemin, A.: The category of ultrametric spaces is isomorphic to the category of complete, atomic, tree-like, and real graduated lattices lat*. Algebra Universalis 50(1), 35–49 (2003)
2. Litak, T., Mikulás, S., Hidders, J.: Relational lattices. In: Höfner, P., et al. (eds.) RAMiCS 2014. LNCS, vol. 8428, pp. 327–343. Springer (2014)
3. Santocanale, L.: A duality for finite lattices (Sep 2009), preprint, available from http://hal.archives-ouvertes.fr/hal-00432113
4. Santocanale, L., Wehrung, F.: Sublattices of associahedra and permutohedra. Advances in Applied Mathematics 51(3), 419–445 (Aug 2013)
5. Santocanale, L., Wehrung, F.: The equational theory of the weak order on finite symmetric groups (May 2014), preprint, available from http://hal.archives-ouvertes.fr/hal-00986148
6. Spight, M., Tropashko, V.: First steps in relational lattice. CoRR abs/cs/0603044 (2006), http://arxiv.org/abs/cs/0603044
7. Spight, M., Tropashko, V.: Relational lattice axioms. CoRR abs/0807.3795 (2008), http://arxiv.org/abs/0807.3795
8. Tropashko, V.: Relational lattice foundation for algebraic logic. CoRR abs/0902.3532 (2009), http://arxiv.org/abs/0902.3532