Tensor products and *-autonomous categories

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The main use of $*$-autonomous categories is in the semantic study of Linear Logic. For this reason, it is thus natural to look for a $*$-autonomous category of locally convex topological vector spaces (TVS). On one hand, Linear Logic inherits its semantics from Linear Algebra, and it is thus natural to build models of Linear Logic from vector spaces [3,5,6,4]. On the other hand, denotational semantics has sought continuous models of computation through Scott domains [9]. Moreover, the infinite nature of the exponential of Linear Logic calls for infinite dimensional spaces, for which topology becomes necessary.

One of the first intuitions that comes to mind when thinking about objects in a $*$-autonomous category is the notion of reflexive vector space, i.e. a a TVS which equals its double dual. When A is a vector space, the transpose $d_A : A \to (A \to \bot) \to \bot$ of the evaluation map $ev_A : (A \to \bot) \times \ A \to \bot$ is exactly the canonical injection of a vector space in its bidual. Then, requiring d_A to be an isomorphism amounts to requiring A to be reflexive. However, the category of reflexive topological vector spaces is not $*$-autonomous, as it is not closed.

Barr [2] constructs two closed subcategories of the category of TVS by restricting to TVS endowed with their weak topology (WTVS) or with their Mackey topology (MTVS), which are both polar topologies. Indeed, if E is a TVS, one can define its dual E' as the space of all continuous linear form on E. Enforcing E with its weak or with its Mackey topology doesn't change E'. The weak topology is exactly the coarsest among the polar topologies, while the Mackey topology is the finest.

Theorem 1 ([2]). *The full subcategories* WTVS *and* MTVS *are $*$-autonomous*[1].

However, these are not categories of reflexive spaces, since the exact definition of reflexivity in functional analysis requires the dual to be endowed with the bounded-open topology.. Indeed, reflexivity is obtained thanks to the use of polar topologies, which are nothing but an internalization of the notion of dual pair, and the dual here is considered with its weak or its Mackey topology.

We showed that one can build a model of Linear Logic [8], one top of the $*$-autonomous category of WTVS. The additive connectives of Linear Logic are interpreted as the cartesian product and coproduct, and the multiplicative conjunction is interpreted as the algebraic tensor product endowed with a well-chosen topology. Non-linear proofs are interpreted as a sequences of monomials.

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[1] This result is straightforward for any polar topology, but Barr obtain this categories as the image of the left adjoint and right adjoint of the functor from the category TVS to the category of pairs

This model offers a semantic interpretation of non-reversibility: positive connectives are interpreted by those constructions on TVS which are not naturally endowed with their weak topology.

Theorem 2 ([8]). *The category of* WTVS *yields a model of Linear Logic.*

One should be able likewise to form a model of Linear Logic made of Mackey spaces, or of any spaces endowed with a specific polar topology.

When constructing the model, one is faced with a choice, as the tensor products could be endowed with several distinct topologies: the *inductive topology*, the *projective topology* and the *injective topology*. See Grothendieck's thesis for their definition [7]. The choice of this topology determines the internal hom-set of the $*$-autonomous category, through the formula $\mathcal{L}(E, F) = (E \otimes F')'$. The hom-set obtained when considering the inductive tensor product, as in the model described above [8], is the usual space of all linear continuous map between E and F.

The study of tensor products led Grothendieck to the notion of *nuclear spaces* [7], which are TVS for which the injective and the projective tensor products correspond. However, every nuclear space which is either Fréchet or (DF)[2] is reflexive, and we obtain in a non-trivial $\star$-autonomous category of TVS. They enjoy moreover remarkable stability properties :

Theorem 3. *There is a full subcategory of the category of Nuclear spaces which is a model Multiplicative Additive Linear Logic.*

Besides, such spaces are very common in functional analysis[3]. Examples of Nuclear and Fréchet include spaces of test functions, spaces of distributions, spaces of differentiable maps on smooth manifold, or spaces of holomorphic maps on analytic manifold. Those functions spaces $\mathcal{F}(V)$ verify kernel theorems, alike the exponential isomorphisms in Seely categories [10] :

$$\mathcal{F}(V)\hat{\otimes}\mathcal{F}(U) = \mathcal{F}(U \times V)$$

References

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[2] A Fréchet space is a complete metrisable TVS. In particular inductive and projective tensor product correspond. A (DF)-space is essentially a pre-dual of Fréchet spaces.

[3] Nuclear spaces are also those spaces for which maps into a Banach are nuclear. This alternative definition hints at a possible relation between our approach and Abramsky, Blute and Panangaden's attempt to generalise the notion of nuclear map into a model of Linear Logic [1].

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