Measurable Preorders and Complexity

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In a recent paper [6], we defined a generic construction of models of the exponential-free fragment of Linear Logic (MALL). These models provide a new framework for the study of computational complexity which allows for the use of techniques and invariants form ergodic theory and operator theory.

The construction of these models is parametrised by a monoid $\mathfrak{m}$ – called a *microcosm* – of measurable maps $\mathbf{X} \rightarrow \mathbf{X}$ on a measured space $\mathbf{X}$. The central notion is that of $\mathfrak{m}$-*graphing*, a generalisation of the notion of directed graphs. Intuitively, a $\mathfrak{m}$-graphing is a graph *realised* on a measured space $\mathbf{X}$: its vertices are measurable subsets of $\mathbf{X}$ and its edges are realised by measurable functions in $\mathfrak{m}$. The resulting framework is combinatorially richer than that of graphs, as for instance two vertices of a graphing may be neither disjoint nor equal. E.g., the graph $\mathbf{2} := \bullet \rightarrow \bullet$ may be realised as $[0, 2/3] \rightarrow [1/3, 1], x \mapsto x + 1/3$. Girard's *Geometry of Interaction* constructions appear as special cases of our construction, and Danos' interpretation [1] of pure lambda-calculus in geometry of interaction therefore provides a representation of all pure lambda-terms as graphings. Moreover, graphings can represent non-deterministic and probabilistic computation [7]. Graphings thus provides a good mathematical abstraction of algorithms, which should adapt to quantum and concurrent computation.

Using realizability techniques, one can define types for these untyped abstract programs. To understand this one can think about the pure lambda-calculus, where it is possible to reconstruct simple types as those sets of terms that behave similarly when put into a particular evaluation context [4]. For instance, the arrow type $A \rightarrow B$ will be given to any program which, given an argument of type A, computes an element of type B. Using the same techniques, one can define types for graphings and show that the resulting structure is a model of MALL. For most measured spaces $\mathbf{X}$ and microcosms $\mathfrak{m}$, one can moreover define sub-exponential connectives. By sub-exponential connectives, we mean here a modality ! (and therefore its dual ?) such that the contraction principle holds, i.e. $!A \multimap !A \otimes !A$. Other principles satisfied by the exponential connectives of linear logic are not required to hold, providing models for bounded logics [5].

We recently proposed [7] to study computational complexity through the lens of these models by studying the class of languages $\mathtt{C}[\mathfrak{m}]$ accepted by elements of type $\mathbf{!W} \multimap \mathbf{Bool}$ in the models defined by $\mathfrak{m}$. As first results, we showed how to obtain a infinite families of models in which the type of predicates char-

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acterizes standard complexity classes[1]. These results leads us to conjecture a correspondence between complexity constraints and microcosms.

To be more precise, we consider the natural notion of *compilation*. A measurable map $m : \mathbf{X} \rightarrow \mathbf{X}$ is *compilable* in a microcosm $\mathfrak{n}$ when there exists a finite partition $X_1, \dots, X_k$ of $\mathbf{X}$ and elements $n_1, \dots, n_k \in \mathfrak{n}$ such that $m_{\upharpoonright X_i} =_{\text{a.e.}} (n_i)_{\upharpoonright X_i}$. This notion will be denoted by $m \prec_{\text{c}} \mathfrak{n}$ and naturally extends to a preorder on microcosms – also denoted $\prec_{\text{c}}$ – which induces the equivalence $\mathfrak{m} \sim_{\text{c}} \mathfrak{n}$.

Theorem 1. *If* $\mathfrak{m} \prec_{\text{c}} \mathfrak{n}$*, then* $\mathtt{C}[\mathfrak{m}] \subset \mathtt{C}[\mathfrak{n}]$.

We conjecture that the converse of Theorem 1 holds, i.e. if $\mathfrak{m} \not\sim_{\text{c}} \mathfrak{n}$ are not equivalent, then $\mathtt{C}[\mathfrak{m}] \neq \mathtt{C}[\mathfrak{n}]$ are distinct. This would provide an equivalence between the problem of classifying complexity classes and that of classifying microcosms – or equivalently graphing algebras[2].

Moreover, the notion of equivalence $\sim_{\text{c}}$ relates to well-studied notions in ergodic theory and von Neumann algebras. If $\mathfrak{m}$ is a microcosm, one can define the preorder $\mathcal{P}_{\mathfrak{m}} = \{(x, y) \mid \exists m_1, \dots, m_k \in \mathfrak{m}, m_1 \circ m_2 \circ \cdots \circ m_k(x) = y\}$. Obviously, if $\mathfrak{m} \sim_{\text{c}} \mathfrak{n}$, the preorders $\mathcal{P}_{\mathfrak{m}}$ and $\mathcal{P}_{\mathfrak{n}}$ are equal (although the converse does not hold). If $\mathfrak{m}$ is a countable group of measure-preserving maps, this preorder is a *measurable equivalence relation* [2], and the notion of ℓ^2-Betti numbers which was extended to this setting by Gaboriau [3] provides a method of separation.

The converse of Theorem 1 would thus offer a proof method for separation results when the microcosms considered are countable groups of measure-preserving maps. It would also advocate for the study of *measurable preorders* and their invariants: if it is natural to assume countability of $\mathfrak{m}$, the restriction to groups and to measure-preserving maps seems too restrictive to capture sufficiently many complexity classes.

References

1. V. Danos. *La Logique Linéaire Appliquée à l'Étude de Divers Processus de Normalisation (principalement du λ-calcul)*. PhD thesis, Paris VII University, 1990.
2. J. Feldman and C. C. Moore. Ergodic equivalence relations, cohomology, and von neumann algebras. I. *Transactions of the AMS*, 234(2):289–324, 1977.
3. D. Gaboriau. Invariants ℓ^2 de relations d'équivalence et de groupes. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 95(93-150):15–28, 2002.
4. C. Riba. Strong normalization as safe interaction. In *LICS 2007 Proceedings*, 2007.
5. T. Seiller. Interaction graphs: Exponentials. *Submitted*, 2014.
6. T. Seiller. Interaction graphs: Graphings. *Submitted*, 2014.
7. T. Seiller. Towards a *Complexity-through-Realizability* theory. *Submitted*, 2015.

[1] Among which regular languages Reg, stochastic languages Stoc, deterministic, non-deterministic and probabilistic logarithmic space classes L, NL, coNL and PL.

[2] The graphing algebra is a generalisation of the group algebra construction; it applies to microcosms which are particular kinds of graphings.