On Regular Congruences of Ordered Semigroups

Boža Tasić

Ryerson University, Toronto ON M5V 0B5, CANADA
btasic@ryerson.ca

Abstract. An ordered semigroup is a structure $\mathbf{S} = \langle S, \cdot, \leq \rangle$ with a binary operation $\cdot$ that is associative and a partial ordering $\leq$ that is compatible with the binary operation. For a given congruence relation θ of the semigroup $\mathbf{S} = \langle S, \cdot \rangle$ the quotient structure $\mathbf{S}/\theta = \langle S/\theta, \odot, \preceq \rangle$ is not in general an ordered semigroup. In this paper we study quotients of ordered semigroups. We first define a special type of congruences, called regular congruences, that will preserve ordering on the quotient structures. We then show that the set of all regular congruences with the ordering $\leq$ is an algebraic lattice. Afterwards, we discuss the link between finitely generated regular congruences and subdirectly irreducible ordered semigroups. At the end we will discuss generalization of these concepts to an arbitrary ordered algebra.

Definition 1. *A partially ordered semigroup (in the remainder of the text po-semigroup) is an ordered triple* $\mathbf{S} = \langle S, \cdot, \leq \rangle$ *such that*

a) $\langle S, \cdot, \rangle$ *is a semigroup,*
b) $\langle S, \leq \rangle$ *is a partially ordered set (briefly poset),*
c) $(x \leq y \ \&\ u \leq v) \rightarrow x \cdot u \leq y \cdot v$ *for all* $x, y, u, v \in S$.

Definition 2. *A congruence relation of the semigroup* $\mathbf{S} = \langle S, \cdot \rangle$ *is an equivalence relation* $\theta \subseteq S^2$ *that satisfies the following compatibility property:*

$$(x\theta y \ \&\ u\theta v) \rightarrow x \cdot u \, \theta \, y \cdot v \quad \textit{for all } x, y, u, v \in S. \tag{CP}$$

Given a congruence relation θ of the semigroup $\mathbf{S} = \langle S, \cdot \rangle$ (or an algebra in general) we define the quotient semigroup (a.k.a the homomorphic image) of $\mathbf{S}$ by θ to be the semigroup $\mathbf{S}/\theta = \langle S/\theta, \odot \rangle$ where

$$x/\theta \odot y/\theta = (x \cdot y)/\theta \quad \text{for all } x, y \in S.$$

The set of all congruences of a semigroup $\mathbf{S} = \langle S, \cdot \rangle$ is denoted by $Con(\mathbf{S})$.

How does one extend the concept of quotients to po-semigroups? In general, for structures that have only operations, i.e., algebras, quotients are defined using congruences. Structures that have both operations and relations are studied in model theory and their quotients are defined as

The algebraic quotient + Relations on the algebraic quotient.

In case of the po-semigroups we have the following definition of the quotient.

Definition 3. *Given a po-semigroup* $\mathbf{S} = \langle S, \cdot, \leq \rangle$ *and* $\theta \in Con\langle S, \cdot \rangle$ *we define the quotient po-semigroup (a.k.a the homomorphic image) of* $\mathbf{S}$ *by* θ *to be the structure* $\mathbf{S}/\theta = \langle S/\theta, \odot, \preceq \rangle$*, where*

a) $\langle S/\theta, \odot \rangle$ *is a quotient semigroup,*
b) The relation $\preceq$ *on* S/θ *is defined by*

$$x/\theta \preceq y/\theta \leftrightarrow (\exists x_1 \in x/\theta)(\exists y_1 \in y/\theta)\, x_1 \leq y_1$$

While quotients preserve identities satisfied by the original algebra, they may not preserve all formulas involving relations satisfied by the original structure. Unfortunately, a quotient of a po-semigroup is not necessarily a po-semigroup. $\preceq$ is reflexive, and $\preceq$ on S/θ is compatible with $\odot$. But neither anti-symmetry nor transitivity is preserved by $\preceq$ in general.

The idea of preserving defining formulas by quotients is very natural, i.e., we would like quotients of po-semigroups to be po-semigroups. Since the relation $\preceq$ introduced in 3 b) does not preserve anti-symmetry and transitivity one may wonder if we can define a quotient of a po-semigroup differently. One way of answering this question is to remove the requirement 3 b), imposed by model theory, and allow for other ordering relations on S/θ. This approach of extending the concept of quotients to po-semigroups, has been taken in [4] and [5]. We first define a special type of congruences.

Definition 4. *Let* $\mathbf{S} = \langle S, \cdot, \leq \rangle$ *be a po-semigroup and* $\theta \in Con\langle S, \cdot \rangle$*. The congruence* θ *is called* **regular** *if there exists an order* $\preceq$ *on* S/θ *such that:*

a) $\mathbf{S}/\theta = \langle S/\theta, \odot, \preceq \rangle$ *is a po-semigroup,*
b) The mapping $\varphi : S \mapsto S/\theta$ *defined by* $\varphi(x) = x/\theta$ *is isotone, i.e.,*

$$x \leq y \to x/\theta \preceq y/\theta.$$

The algebraic part $\langle S/\theta, \odot \rangle$ of the quotient po-semigroup $\mathbf{S}/\theta$ is simply the semigroup quotient of $\langle S, \cdot \rangle$ by θ. The relational part $\langle S/\theta, \preceq \rangle$ of the quotient po-semigroup $\mathbf{S}/\theta$ is the partially ordered set and the multiplication $\odot$ is compatible with $\preceq$, i.e., (CPQ) is satisfied.

It turns out that the above notion of regularity can be expressed using a special property of θ. The following theorem was proved in [5, pg 168 Thm 3.4], and [1, pg 42 Thm 6.1].

Theorem 1. *Let* $\mathbf{S} = \langle S, \cdot, \leq \rangle$ *be a po-semigroup. A congruence* $\theta \in Con\langle S, \cdot \rangle$ *is regular if and only if it satisfies the property: for every* $n \in \mathbb{N}$ *and every* $a_1, \ldots, a_{2n} \in S$

$$a_1 \,\theta\, a_2 \leq a_3 \,\theta\, a_4 \leq \ldots\ldots a_{2n-1} \,\theta\, a_{2n} \leq a_1 \;\to\; \{a_1, \ldots, a_{2n}\}^2 \subseteq \theta. \tag{1}$$

Given a po-semigroup $\mathbf{S} = \langle S, \cdot, \leq \rangle$ we denote by $RCon(S)$ the set of all regular congruences on $\mathbf{S}$. The following result was proved in [5].

Theorem 2. *For a given po-semigroup* $\mathbf{S} = \langle S, \cdot, \leq \rangle$*, the set of all regular congruences is a complete lattice with respect to the inclusion as the ordering. We denote it by* $\mathbf{RCon\,S} = \langle RConS, \subseteq \rangle$*.*

We will show that the lattice $\mathbf{RCon\,S}$ in Theorem 2 is actually an algebraic lattice.

Theorem 3. *For a po-semigroup* $\mathbf{S} = \langle S, \cdot, \leq \rangle$*, there is an algebraic closure operator* Θ *on* $S \times S$ *such that the closed subsets of* $S \times S$ *are precisely the regular congruences on* $\mathbf{S}$*. Hence* $\mathbf{RCon\,S} = \langle RCon(S), \subseteq \rangle$ *is an algebraic lattice.*

The compact members of $\mathbf{RCon\,S}$ are, by [2], the finitely generated members $\Theta((a_1, b_1), \dots (a_n, b_n))$ of $\mathbf{RCon\,S}$.

We will also discuss the link between finitely generated regular congruences and subdirectly irreducible ordered semigroups. At the end we will discuss generalization of these concepts to an arbitrary ordered algebra.

References

1. Blyth, T. S., Janowitz M.F.: Residuation Theory. Pergamon Press, Oxford, (1972)
2. Burris, S., Sankappanavar, H. P.: A Course in universal algebra. Springer Verlag, New York (1981)
3. Bloom, S. L.: Varieties of Ordered Algebras. Journal of Computer and System Sciences **13**, 200–212 (1976)
4. Kehayopulu, N., Tsingelis M.: On Subdirectly Irreducible Ordered Semigroups. Semigroup Forum Vol. **50**, 161–177 (1995)
5. Xiang-Yun, X.: On Regular, Strongly Regular Congruences on Ordered Semigroups. Semigroup Forum Vol. **61**, 159–178 (2000)