Products in the category of forests and p-morphisms via Delannoy paths on Cartesian products

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In [5], the authors introduce a technique to compute finite coproducts of finite Gödel algebras, *i.e.* Heyting algebras satisfying the prelinearity axiom $(\alpha \to \beta) \vee (\beta \to \alpha)$. To do so, they investigate the product in the category opposite to finite Gödel algebras: the category of forests and open order-preserving maps, *alias* p-morphisms, which we denote by F. (A forest is a partially ordered set F such that, for every $x \in F$, the set of lower bounds of x forms a chain, when endowed with the order inherited from F.) To achieve their result, the authors make use of ordered partitions of finite sets and of a specific operation — called *merged-shuffle* — on ordered partitions. In [1, Section 4.2], the authors present an alternative, recursive construction of finite products in the category of forests and open order-preserving maps.

In the present work we introduce a further construction of the same finite products, based on products of posets along with a generalization of the combinatorial notion of *Delannoy path*. The new and most interesting aspect of our construction is that, dually, it uncovers a key relationship between the coproducts of finite Gödel algebras and the coproducts in the category of finite distributive lattices. Our main result explains the former coproducts in terms of a construction on the latter; the construction itself is currently best understood via duality using a generalisation of the Delannoy paths.

Classically, a *Delannoy path* (see [4, p.80]) is a path on the first integer quadrant $\mathbb{N}^2 \subseteq \mathbb{Z}^2$ that starts from the origin and only uses northward, eastward, and north-eastward steps. We begin by generalizing the notion of Delannoy path to Cartesian products of finite posets. A (finite) *path* on a poset P is a sequence $< p_1, p_2, \ldots, p_h >$ of elements of P such that $p_i < p_j$ whenever $i < j$. (A path on P is therefore the same thing as a chain of P.) For each $i \in \{1, \ldots, n-1\}$, the pair p_i, p_{i+1} is called a *step* of the path. Given a poset P, and two elements $p, q \in P$, we write $p \lhd q$ to indicate that q covers p in P, that is, $p < q$ and for every $s \in P$, if $p \leq s \leq q$, then either $s = p$ or $s = q$.

In [3], the notion of Delannoy path has been extended to finite products of chains. The following generalization is perhaps less obvious.

Definition 1. *Let $P_1, P_2, \ldots, P_n$ be posets, and let $P = P_1 \times P_2 \times \cdots \times P_n$ be their (Cartesian) product. Let $< p_1, p_2, \ldots, p_h >$ be a path on P. The step from $p_i = (p_{i,1}, p_{i,2}, \ldots, p_{i,n})$ to $p_{i+1} = (p_{i+1,1}, p_{i+1,2}, \ldots, p_{i+1,n})$ is a* Delannoy step*, written $p_i \prec p_{i+1}$, if and only if there exists $k \in \{1, \ldots, n\}$ such that $p_{i,k} \neq p_{i+1,k}$, and for each $j \in \{1, \ldots, n\}$, $p_{i,j} = p_{i+1,j}$, or $p_{i,j} \lhd p_{i+1,j}$. The path $< p_1, p_2, \ldots, p_h >$ on P is a* Delannoy path *if and only if p_1 is a minimal element of P, and for each $i \in \{1, \ldots, n-1\}$, $p_i \prec p_{i+1}$.*

A Delannoy path on P is thus a sequence of Delannoy steps starting from a minimal element of P. Delannoy paths on a poset $P = P_1 \times \cdots \times P_n$ can be partially ordered by $< q_1, \ldots, q_m > \leq < p_1, \ldots, p_h >$ if and only if $m \leq h$ and $q_i = p_i$ for each $i \in \{1, \ldots, m\}$. We denote by $\mathcal{D}(P_1, \ldots, P_n)$ the poset of all Delannoy paths on P. Clearly, $\mathcal{D}(P_1, \ldots, P_n)$ is a forest.

Definition 2 (Product). *Let F and G be forests. We call $F \times_{\mathsf{F}} G = \mathcal{D}(F, G)$ the* product *of F and G.*

Definition 3 (Projections). *Let F and G be forests, let $\{f_1, \ldots, f_m\}$ and $\{g_1, \ldots, g_n\}$ be the underlying sets of F and G, respectively, and let $D = F \times_{\mathsf{F}} G$. We define a function $\pi_F : D \to F$ such that for each Delannoy path $d \in D$, with $d =< (f_i, g_j), \ldots, (f_h, g_k) >$, $\pi_F(d) = f_h$. Analogously, we define a function $\pi_G : D \to G$ such that $\pi_G(d) = g_k$.*

Our main result follows.

Theorem 1. *Let F and G be forests. Then*

$$F \xleftarrow{\pi_F} F \times_{\mathsf{F}} G \xrightarrow{\pi_G} G$$

is the product of F and G in the category F.

Remark. We point out the parallel with [2], where the authors use the forest of *all* paths on a finite poset to construct the Gödel algebra freely generated by a finite distributive lattice.

References

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