Orthogonal relational systems

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Extended Abstract

It is superfluous to recall how important binary relational systems are for the whole of mathematics. The study of binary relations traces back to the work of J. Riguet [15], and a first attempt to provide an algebraic theory of relational systems is due to Mal'cev [13]. A general investigation of quotients and homomorphisms of relational systems can be found in [3], where seminal notions from [1] are developed. A leading motivation for our discussion stems from the theory of *semilattices*. In fact, semilattices can be equivalently presented as ordered sets as well as groupoids. This approach was widen to ordered sets whose ordering is directed. In this case the resulting groupoid needs not be, in general, a semilattice, but a *directoid* [4]. We will see that many features of a relational system $\mathbf{A} = \langle A, R \rangle$ can be captured by means of the associated groupoid. Reflexivity, symmetry, transitivity or antisymmetry of R can be equationally or quasi-equationally characterized in the groupoid [5, 6].

The concept of orthogonal poset was first considered in [2], where an algebraic characterization of the system through the associated groupoid with involution is presented. Orthogonal posets represent an important tool in the investigation of logics from quantum mechanics because they are recognized as quantum structures. In [7] this method was generalized to cover the case of ordered sets with antitone involution. These ideas motivated us to extend the approach to general algebraic systems with involution and distinguished elements. The aim of the present talk is to develop this theory generalizing the concept of orthogonal poset with that of orthogonal relational system.

In the first part of the talk we introduce the notions of orthogonal relational system and orthogonal groupoids and explain how the two concepts are mutually related. Afterwards we show that the class of orthogonal groupoids enjoys several desirable algebraic properties. In particular, orthogonal groupoids can be treated in the general setting of the theory of "Boolean-like" algebras [16], which allows to prove a decomposition theorem for a variety of orthogonal groupoids. Finally we show that the class of orthogonal groupoids enjoys the strong amalgamation property.

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