Generalizing the clone–coclone Galois connection

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Recall that a set of operations $f\colon B^n \to B$ on a base set B is a *clone* if it is closed under composition, and contains all projections; essentially, clones are algebras that forgot their signature. Apart from universal algebra, clones have numerous applications in computer science and logic, not the least because of the complete classification of clones on $B = \{0,1\}$ due to Post [6]. An important tool in their study is the clone–coclone Galois connection [5, 3, 4]: a clone $\mathcal{C}$ is completely described (at least for finite B) by its set of *invariants* $\mathrm{Inv}(\mathcal{C})$, consisting of relations $R \subseteq B^k$ preserved by all operations $f \in \mathcal{C}$. Classes of the form $\mathrm{Inv}(\mathcal{C})$ are exactly the *coclones*: classes of relations closed under primitive positive definitions.

In this talk, we will look at ways how to extend this Galois connection to more broad classes of "operations". Our motivating example are reversible transformations, i.e., permutations $f\colon B^n \to B^n$, which are of fundamental importance in quantum computing; classes of reversible transformations on $\{0,1\}$ closed under a handful of natural closure conditions including composition were recently classified by Aaronson, Grier and Schaeffer [2]; cf. [1]. A typical example is the class of all reversible transformations preserving Hamming weight modulo m, for some constant m; this shows that we need invariants of a more complicated nature than in the classical case, as permutations can "count".

In general, we will consider classes of *partial multifunctions* $f\colon B^n \Rightarrow B^m$ (which just means $f \subseteq B^n \times B^m$). We will see that the appropriate notion of invariants in this context is given by "weight functions" $w\colon B^k \to M$, where M is a *partially ordered monoid*: then f preserves w-weight if for all matrices $(a_i^j)_{i<n}^{j<k} \in B^{n\times k}$, $(b_i^j)_{i<m}^{j<k} \in B^{m\times k}$ such that $f(a^j) = b^j$, we have

$$\prod_{i<n} w(a_i) \le \prod_{i<m} w(b_i).$$

We will describe closed classes of partial multifunctions and of weight functions in the induced Galois connection, and discuss its variants under additional closure conditions.

References

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