Finitely presented MV-algebras, unital ℓ-groups and rational polyhedra–together

in memoriam Franco Montagna

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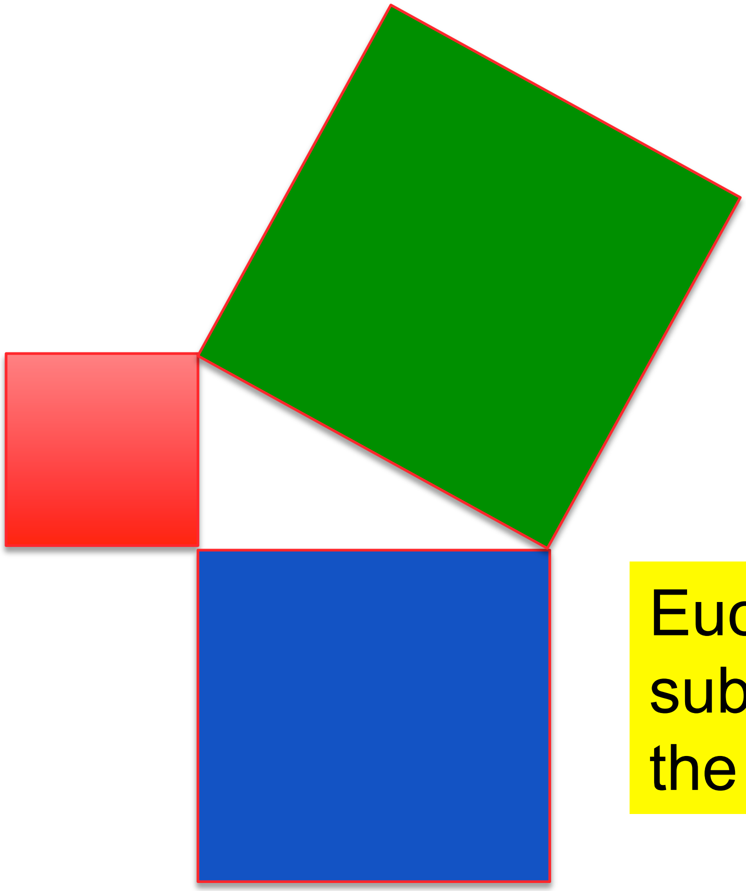
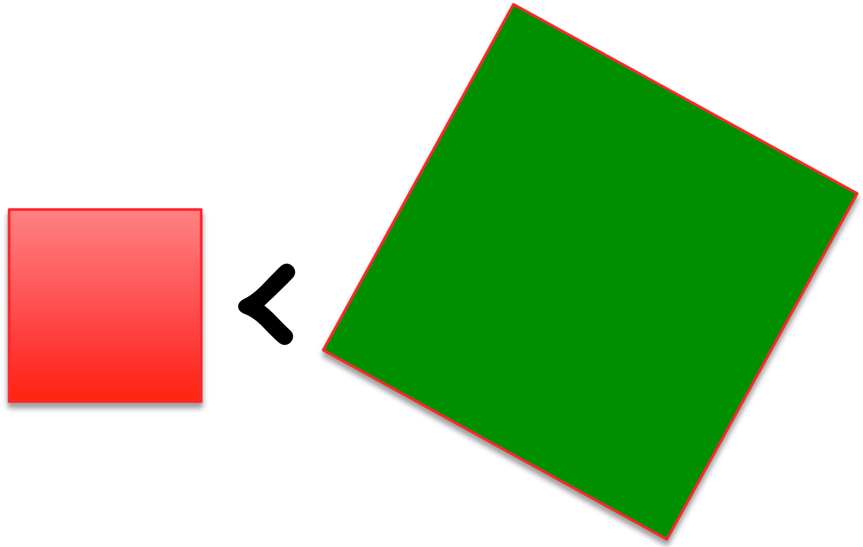
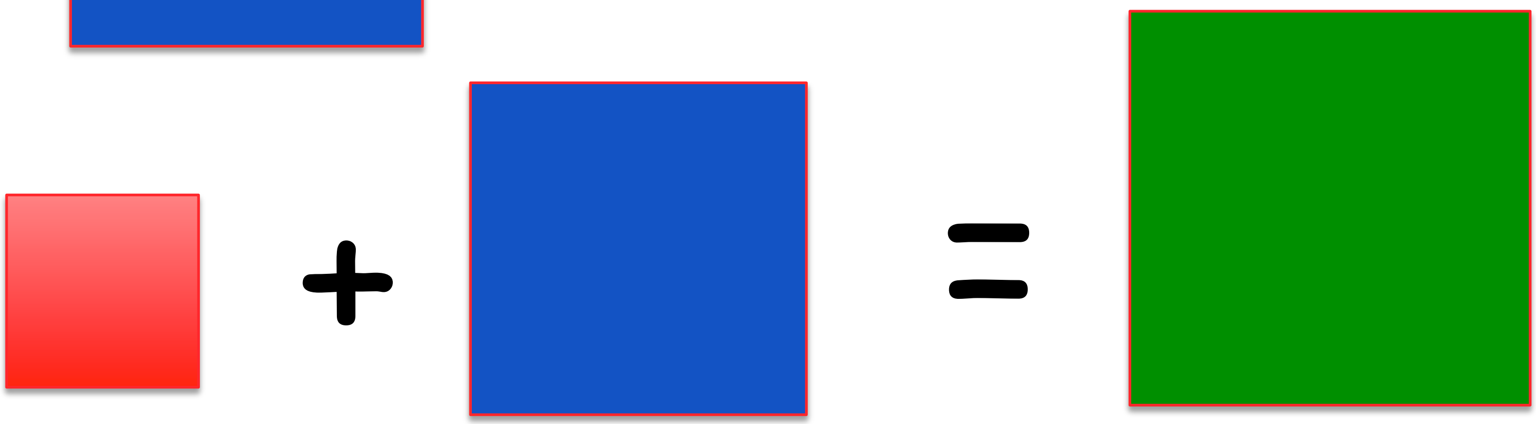




Euclidean magnitudes can be summed, subtracted and compared. The unit has the archimedean property



euclidean magnitudes, *l*-groups, rational polyhedra

since Hölder’s times, addition, subtraction and comparison of magnitudes are carried on in **totally ordered abelian groups**

lattice ordered abelian groups **(l-groups)** describe magnitude-valued functions defined on compact spaces

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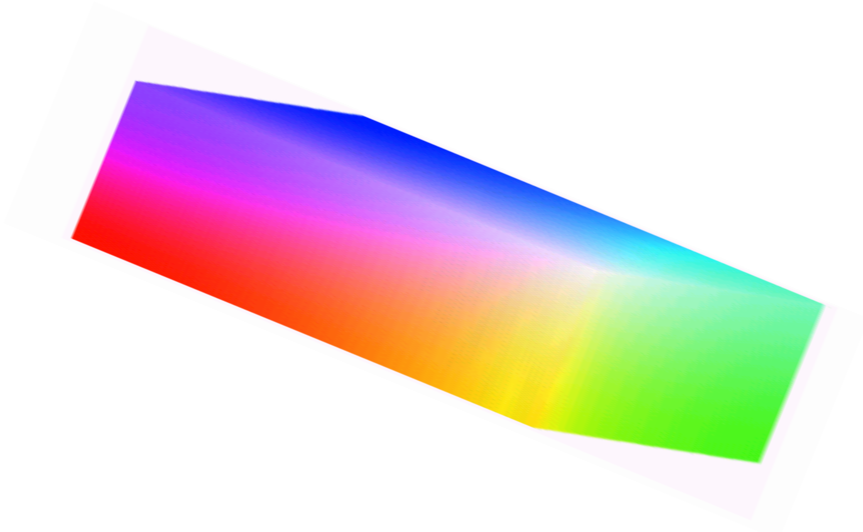
what about the unit?

Since the archimedean property of the unit is undefinable even in first-order logic, unital *l*-groups have been largely neglected

making unital l-groups an equational class (for all conceivable purposes)



these equations contain nice topological, algebraic, geometric, arithmetic, logic-algorithmic structure

$$(x \oplus y) \oplus z = x \oplus (y \oplus z)$$

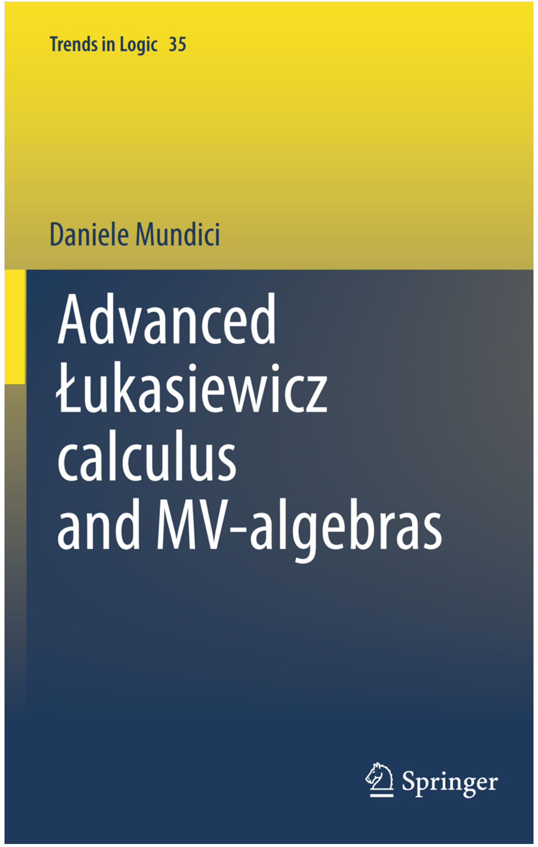
$$x \oplus y = y \oplus x$$

$$x \oplus 0 = x$$

$$\neg\neg x = x$$

$$x \oplus \neg 0 = \neg 0$$

$$\neg(y \oplus \neg x) \oplus y = \neg(x \oplus \neg y) \oplus x$$



these axioms are a reformulation of the time-honored **Lukasiewicz axioms** for his infinite-valued calculus (Actually, the commutativity axiom follows from the others)

MV-algebras ≈ unital *l*-groups

THEOREM (D.M., 1986, J.Functional Analysis) *There is a categorical equivalence* Γ *between unital l-groups and MV-algebras*.

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EXPORT 1: Since MV-algebras are defined by equations, via Γ we can speak of **free objects and finitely presented unital *l*-groups**, as the correspondents of free and finitely presented MV-algebras

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EXPORT 1: Since MV-algebras are defined by equations, via Γ we can speak of **free objects and finitely presented unital *l*-groups**, as the correspondents of free and finitely presented MV-algebras

EXPORT 2: Since MV-algebras are the Lindenbaum algebras the Lukasiewicz infinite-valued calculus, they export to unital *l*-groups their own **natural built-in deductive algorithmic structure**

the category $\mathscr{K}$ of finitely presented **unital** *l*-groups makes perfect sense

THEOREM *For a unital l–group (G,u) the following are equivalent:*

$\Gamma(G,u)=A$ *for some finitely presented MV-algebra A*

(G,u) is finitely presentable as a pointed l-group

The covariant hom-functor hom((G,u), –) : $\mathscr{K} \to$ *Set preserves directed colimits*

[V. Marra, L.Spada, Two isomorphism criteria for directed colimits, arXiv 1312.0432]

the unit makes the difference

THEOREM (Baker-Beynon) *An l-group G is finitely generated projective iff it is finitely presented*

FACT (Folklore) *Every finitely generated projective unital l-group G is finitely presented—* ***but the converse fails***

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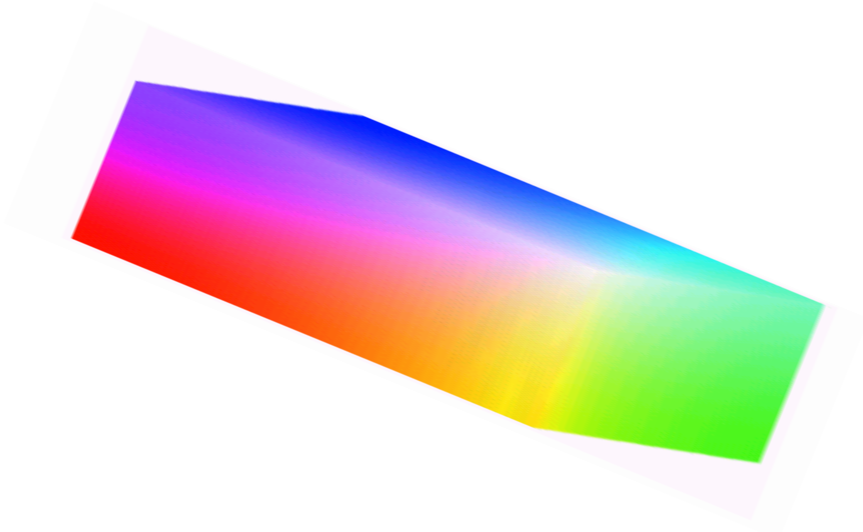
Actually, the characterization of finitely generated projective **unital** *l*-groups is a nice tour de force in algebraic topology.

L.M.Cabrer, D.M., Communications in Contemporary Mathematics 14.3 (2012)
D.M., Combinatorics, Probability and Computing, 23 (2014)
L.M.Cabrer, arXiv 1405.7118 (where the characterization is finally achieved)

finitely presented MV-algebras and unital ℓ-groups are also dually equivalent to a category of rational polyhedra

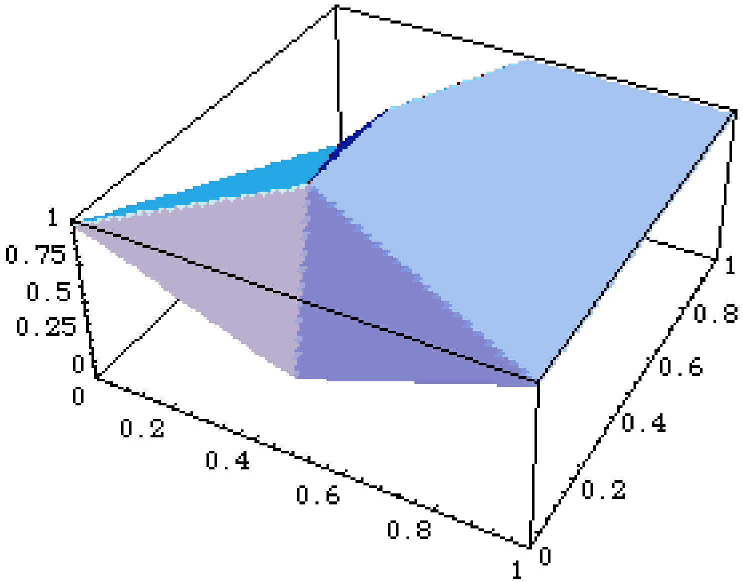


duality in action:
a Lukasiewicz formula ϕ (says very little)

$(x \,\&\, (x \vee y)) \vee ((x \to y) \,\&\, (y \to x))$

legenda: $a\&b = \neg(\neg a \oplus \neg b)$, $a \to b = \neg a \oplus b$, $a \vee b = \neg(\neg a \oplus b) \oplus b$

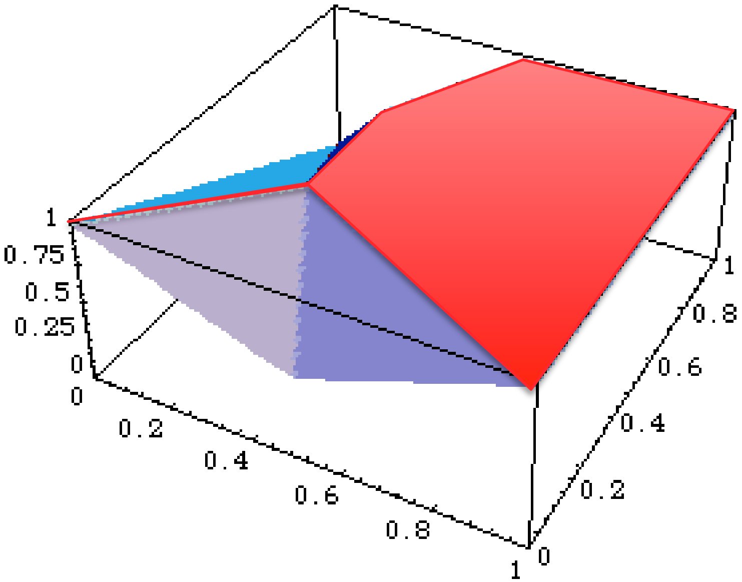
the MV-term ϕ codes a McNaughton map f_ϕ in the free MV-algebra $FREE_n$



(x & (x V y)) V ((x→y) & (y→x))

legenda: a&b=¬(¬a⊕¬b), a→b=¬a⊕b, aVb=¬(¬a⊕b)⊕b

the model-set $Mod(\phi) = f_{\phi}^{-1}(1)$ is a rational polyhedron



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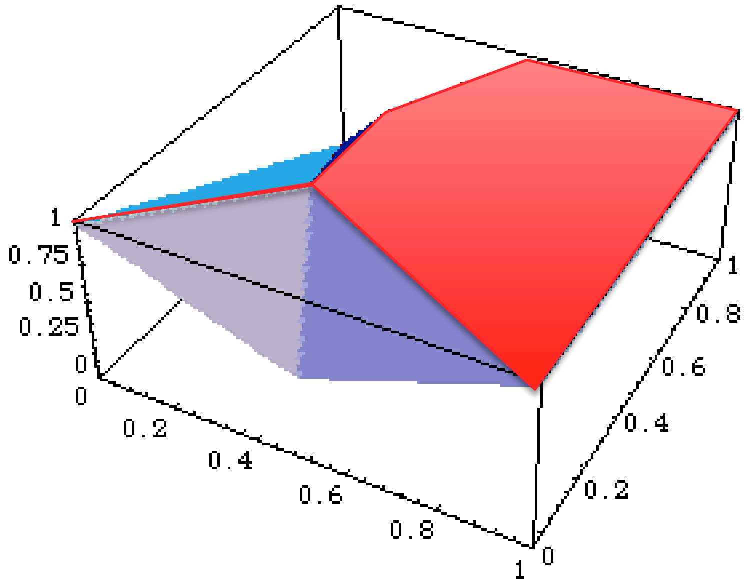
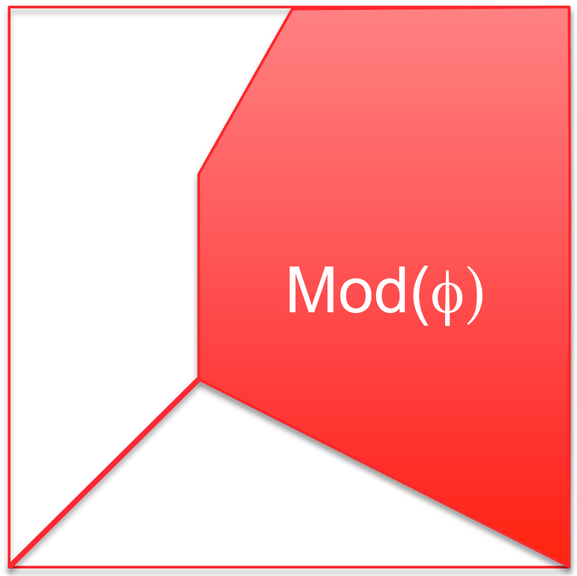
the Lindenbaum algebra L_ϕ is finitely presented by ϕ





$(x \mathbin{\&} (x \vee y)) \vee ((x \rightarrow y) \mathbin{\&} (y \rightarrow x))$

L_ϕ is obtained by restricting to Mod(ϕ) all maps of $FREE_n$

$L_\phi = \mathcal{M}(\mathrm{Mod}(\phi))$ = the McNaughton functions over Mod(ϕ)

DEFINITION An MV-algebra Q is finitely presented if it is the quotient $Q = FREEMV_n/\langle q \rangle$ by some principal ideal $J=\langle q \rangle$, where $q \in FREEMV_n$

C.C.CHANG: MV= HSP[0,1]

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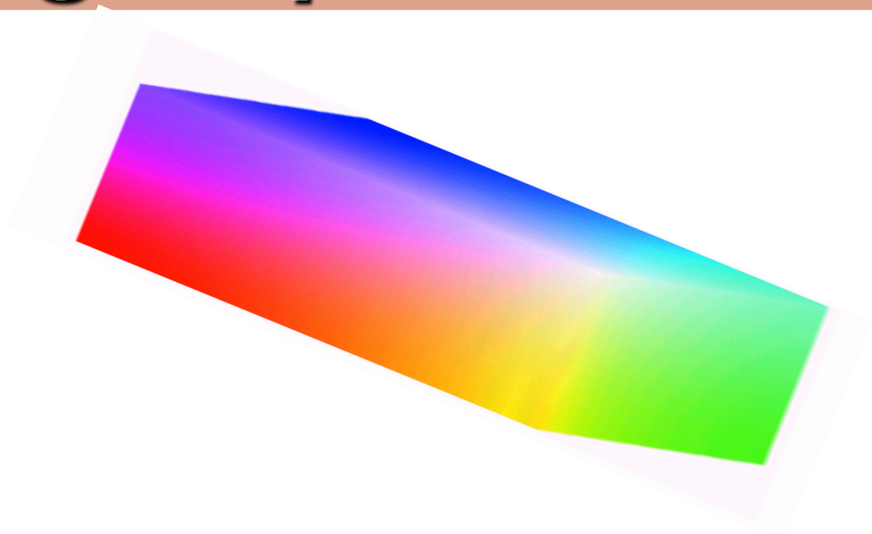
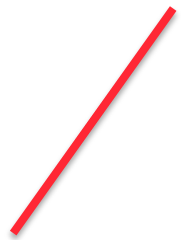
THUS: $Q=FREEMV_n/\langle q \rangle \approx \mathcal{M}(q^{-1}(1)) = FREEMV_n|q^{-1}(1)$ = the restrictions to the rational polyhedron $q^{-1}(1)$ of $q \in FREEMV_n$

introducing the arrows between rational polyhedra, in their duality with finitely presented MV-algebras and unital *l*-groups



the category $\mathcal{P}$ of **resource-aware** polyhedra

this is a segment (geometry is the art of imagining figures independently of their coordinates)



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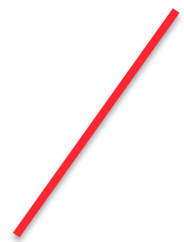
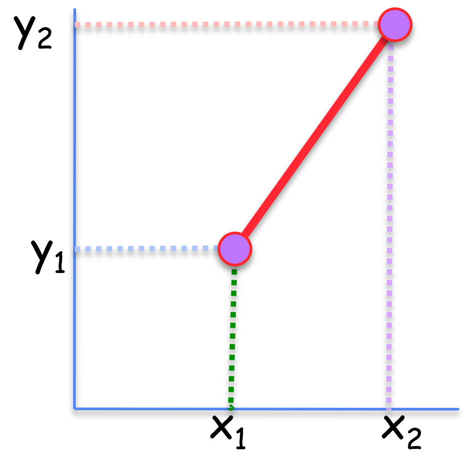


this is a "rational segment" (the coordinates of its vertices are explicitly specified by **rational** numbers)



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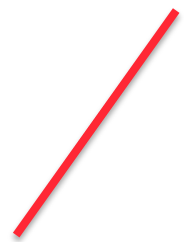

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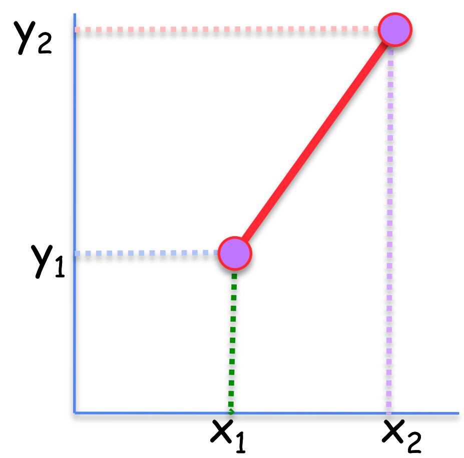
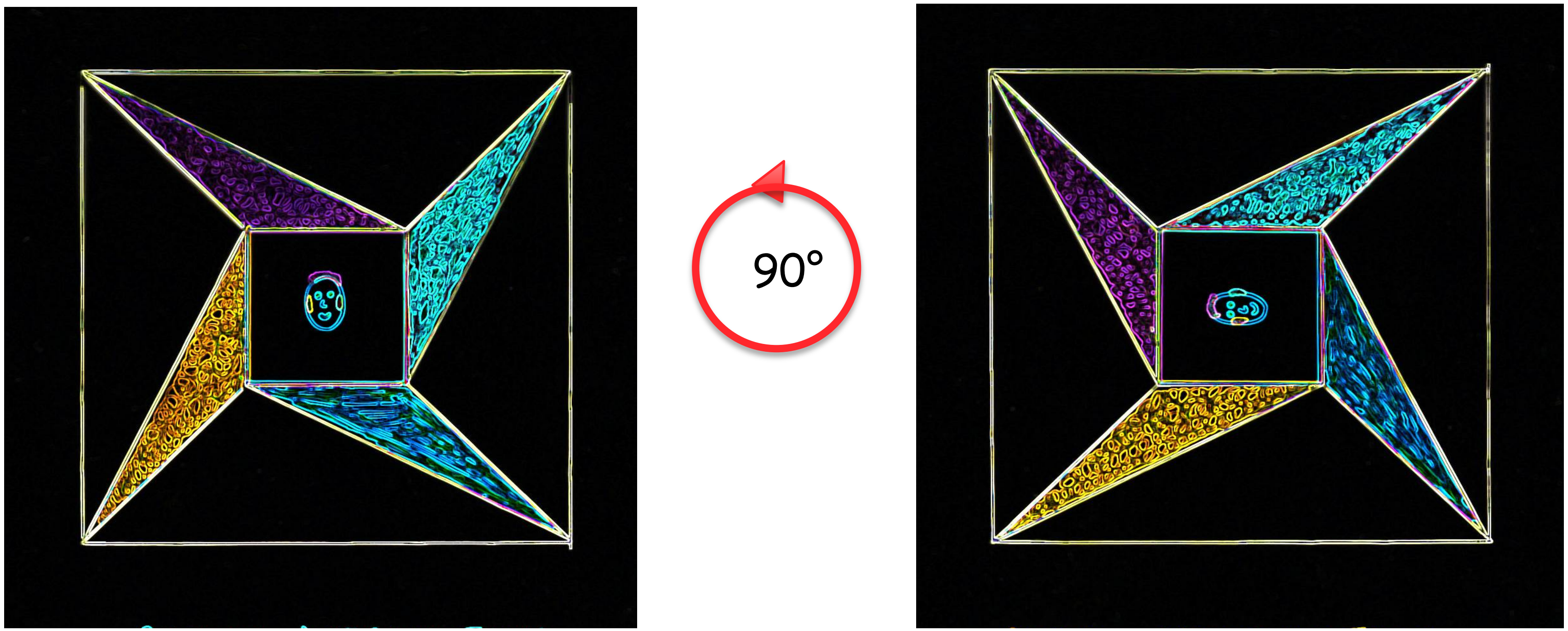


invertible arrows in $\mathcal{P}$ are known as ***Z-homeomorphisms***

Z-homeomorphisms preserve the amount of information needed to specify rational points by definition, a **Z-homeomorphism** is a PL-homeomorphism that preserves least common denominators of the coordinates of rational points.



G. Panti's famous **Z**-homeomorphism A of the unit square onto itself (answering a problem of G-C. Rota)

Z-homeomorphism and the affine group on **Z**

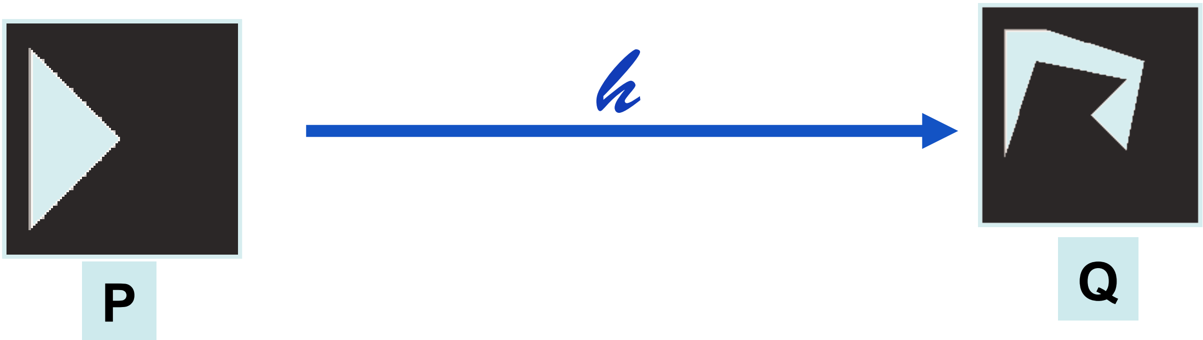
Z-homeomorphisms generate a **new geometry of rational polyhedra**, as isometries do in Euclidean geometry

Since **Z**-homeomorphisms preserve the lattice $\mathbf{Z}^n$ of integer points in $\mathbf{R}^n$, then a **linear Z-homeomorphism** is a member of the n-dimensional **affine group over the integers** $\mathbf{A}_n$

Z-homeomorphism = continuous $\mathbf{A}_n$-equidissection

arrows in this duality: Z-maps

DEFINITION A **Z-map** is a PL-map with integer coefficients



Z-homeomorphism *h* of rational polyhedra P,Q in n-space
=denominator preserving rational PL-homeomorphism h
=invertible **Z**-map *h* whose inverse is also a **Z**-map
=continuous $\mathbf{A}_n$-equidissection *h*, A_n=n-dimensional affine group on **Z**

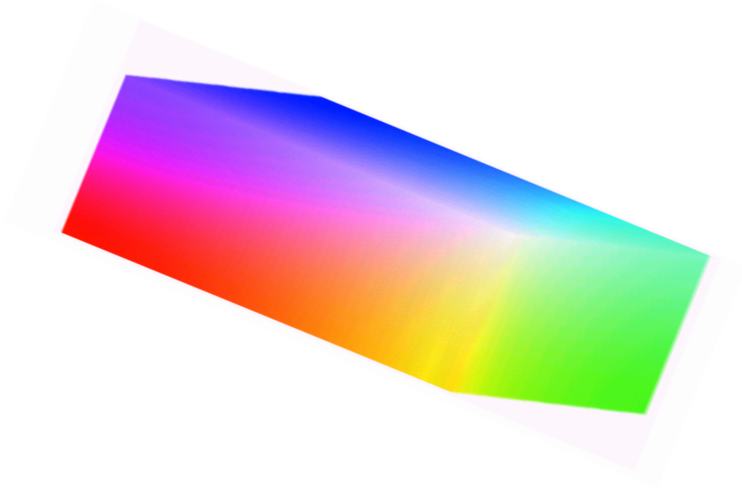
the folklore duality between finitely presented algebras and rational polyhedra with **Z**-maps

OBJECTS: *The map $P \rightarrow \mathcal{M}(P)$ sending each rational polyhedron $P \subseteq [0,1]^n$ to the MV-algebra of McNaughton functions over P, yields a duality between rational polyhedra and finitely presented MV-algebras (≈unital l-groups).*

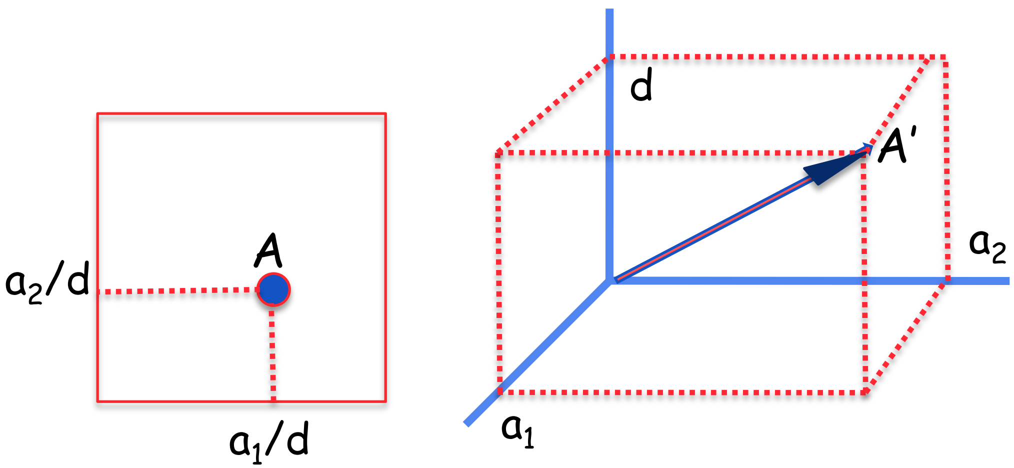
ARROWS: *Every* **Z**-*map $f{:}Q \rightarrow P$ determines the homomorphism $f'{:}\mathcal{M}(P) \rightarrow \mathcal{M}(Q)$ that transforms each McNaughton function g of $\mathcal{M}(P)$ into the composite function $g \circ f$ of $\mathcal{M}(Q)$. Every homomorphism of $\mathcal{M}(P)$ into $\mathcal{M}(Q)$ arises in this way.*

Key algebraic-geometric notions arising from this duality:

1. The homogeneous correspondents of rational points and simplexes

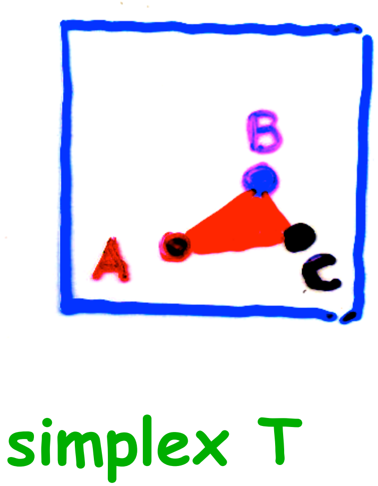


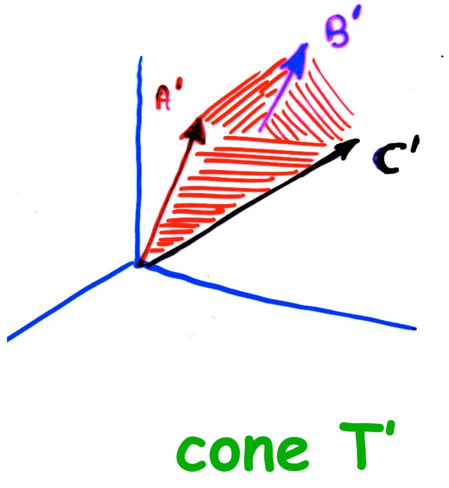
the **homogeneous integer coordinates** of a rational point in $\mathbf{Q}^n$ yield its homogeneous correspondent in $\mathbf{Z}^n$



- let $A = (a_1,\dots,a_n)$ be a rational point in $\mathbf{R}^n$
- the **denominator** of A is the least common denominator d of the coordinates of A
- then $d \cdot (a_1,\dots,a_n,1)$ is an integer vector A' in $\mathbf{Z}^{n+1}$
- A' is said to be the **homogeneous correspondent** of A

the homogeneous correspondent of a simplex





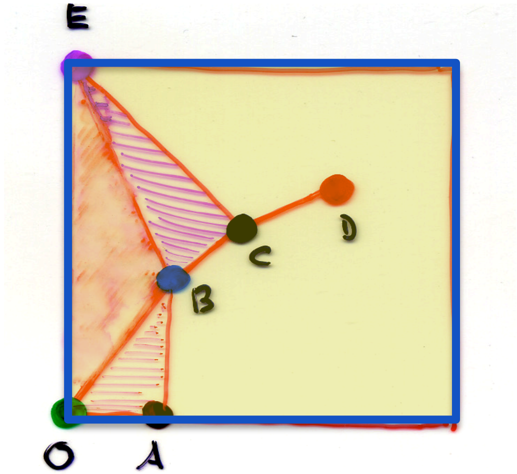
the cone T' is the *positive span* pos(A',B',C') in $\mathbf{R}^3$ of the homogeneous correspondents A'B'C' of the vertices of a simplex T

A'B'C' are the *generating vectors* of T'

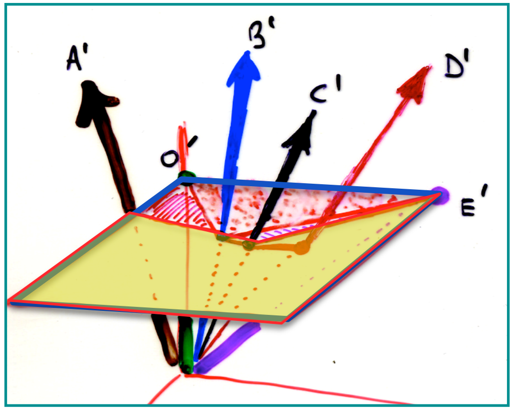
T = conv(V_0,V_1,...,V_k), a k-simplex with rational vertices

T' = pos(V'_0,V'_1,,...V'_k), a k-dimensional cone with generators V'_i

the homogeneous correspondent of a simplicial complex

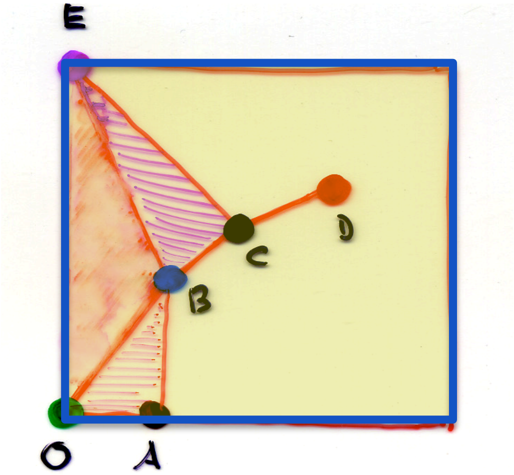


A simplicial complex C with rational vertices in $\mathbf{R}^2$ (any two faces intersect in a common face)



Its corresponding **fan** in $\mathbf{R}^3$, a complex of cones with rational vertices given by the homogeneous correspondents of the vertices of C

the homogeneous correspondent of a simplicial complex



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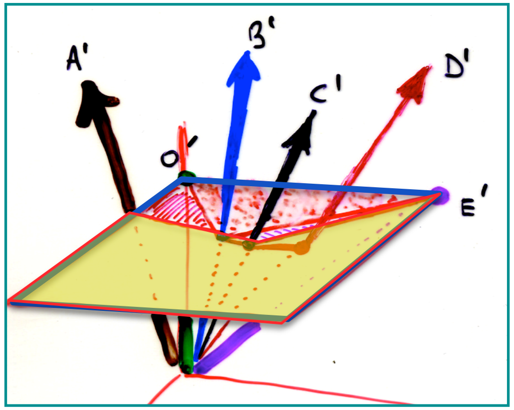

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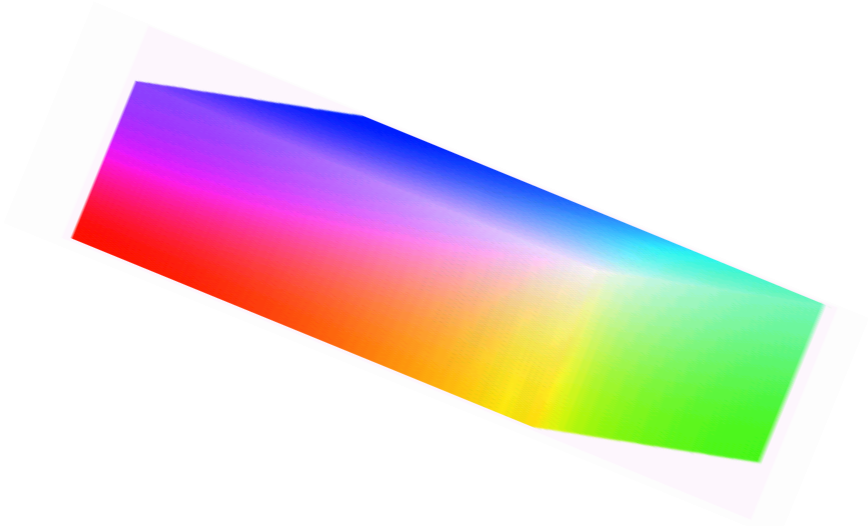
Fans classify toric varieties

Key algebraic-geometric notions arising from this duality:

2. Regular simplicial complexes

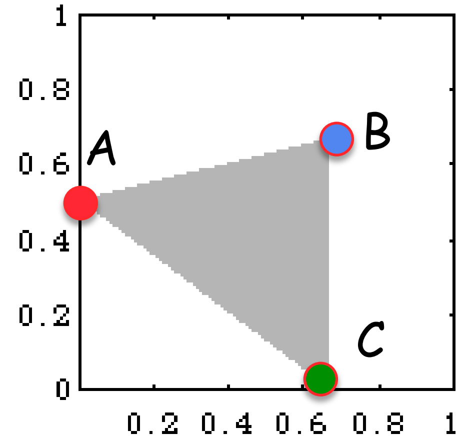


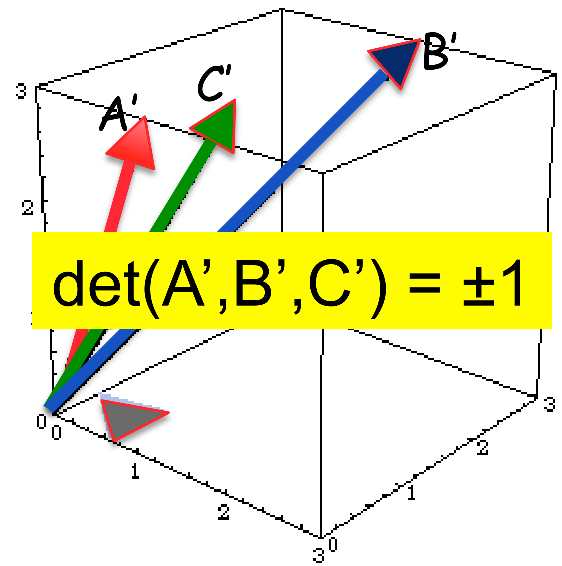
regular simplex

DEFINITION A simplex *T* is

regular
nonsingular, or
unimodular, or

if the set of homogeneous correspondents of its vertices can be completed to a matrix with determinant ± 1



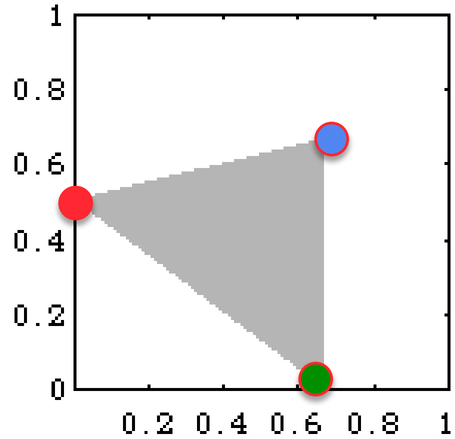


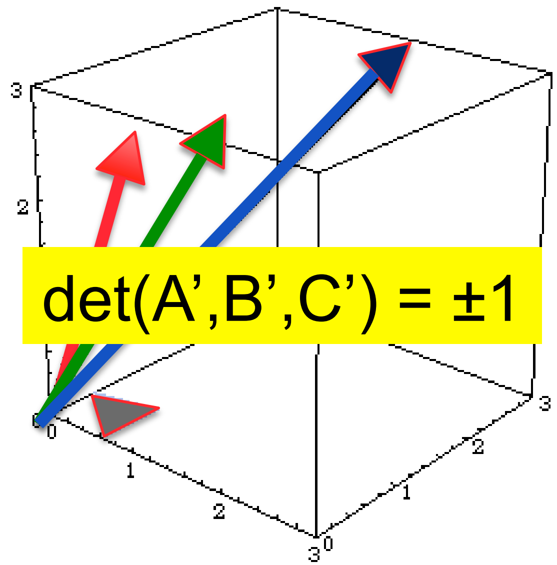
equivalent reformulations of regularity

(from algebra) the homogeneous correspondents are part of a basis in the free abelian group $\mathbf{Z}^{n+1}$

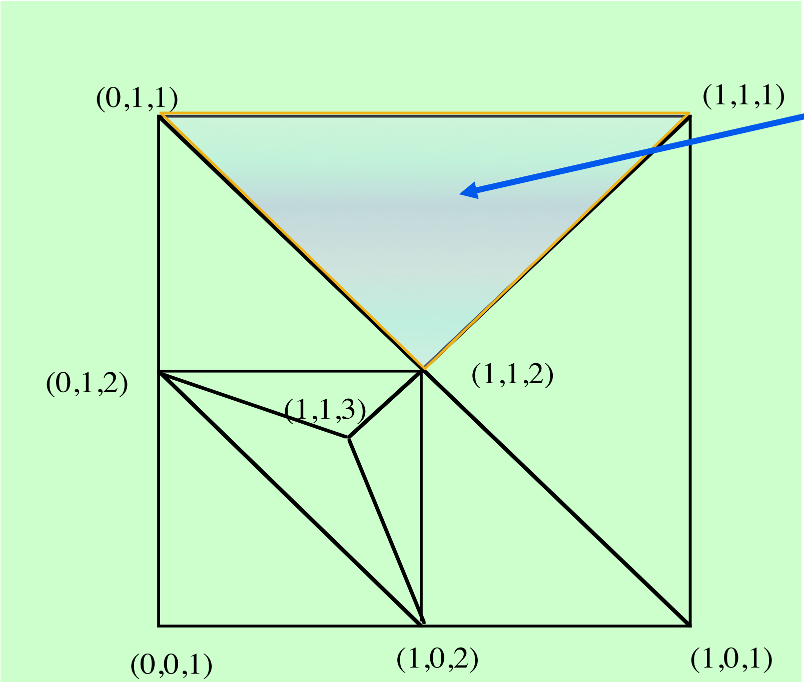
(from the geometry of numbers) the half-open parallelepiped determined by the homogeneous correspondents does not contain any nonzero integer point

(from measure theory) the half-open parallelepiped determined by the homogeneous correspondents has unit volume





Hironaka's regular triangulation of $[0,1]^2$



the homogeneous coordinates of this triangle give the unimodular matrix $M = ((1,1,2),(1,1,1),(0,1,1))$

similarly, every simplex in this triangulation is regular

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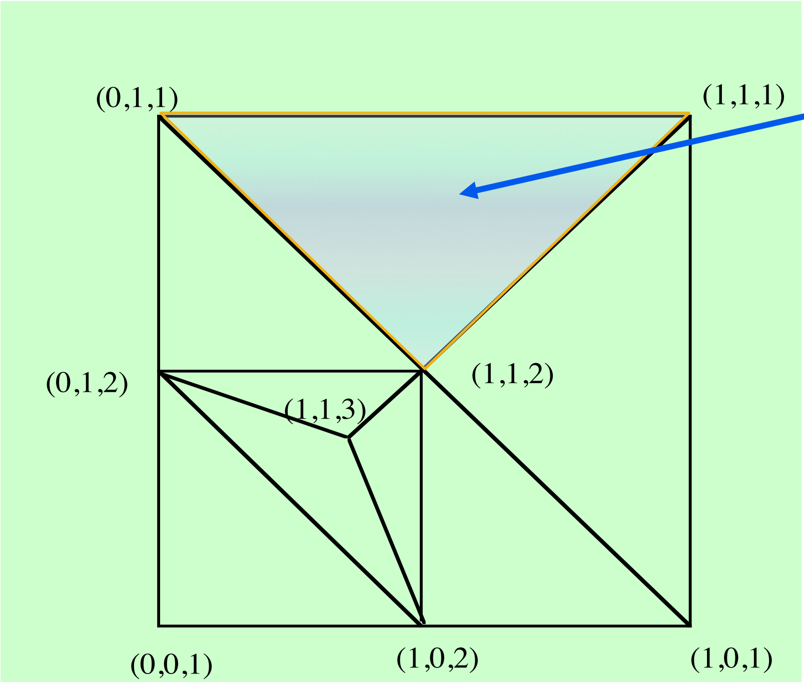

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similarly, every simplex in this triangulation is regular

regular simplexes have found recent applications in the classification of orbits under the affine groups over the integers

[see L.Cabrer, D.Mundici, *Ergodic Theory and Dynamical Systems*, to appear, arXiv 1403.3827]

affine/homogeneous (at the end of the day)

rational point ⇔ integer vector

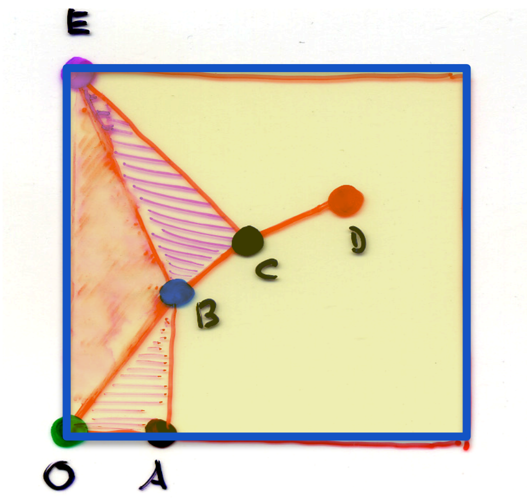
rational simplex ⇔ rational cone

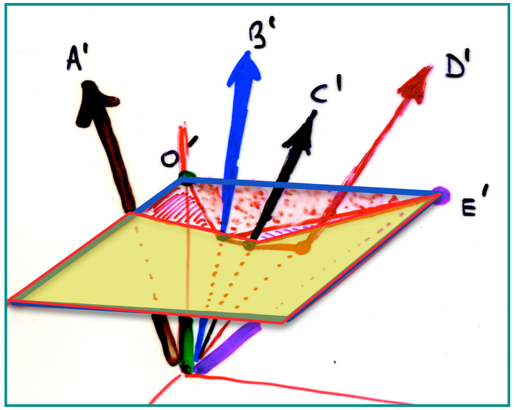
regular simplex ⇔ regular cone

vertices of simplex ⇔ generators of cone

simplicial complex ⇔ simplicial fan

regular complex ⇔ nonsingular fan
⇔ smooth toric variety

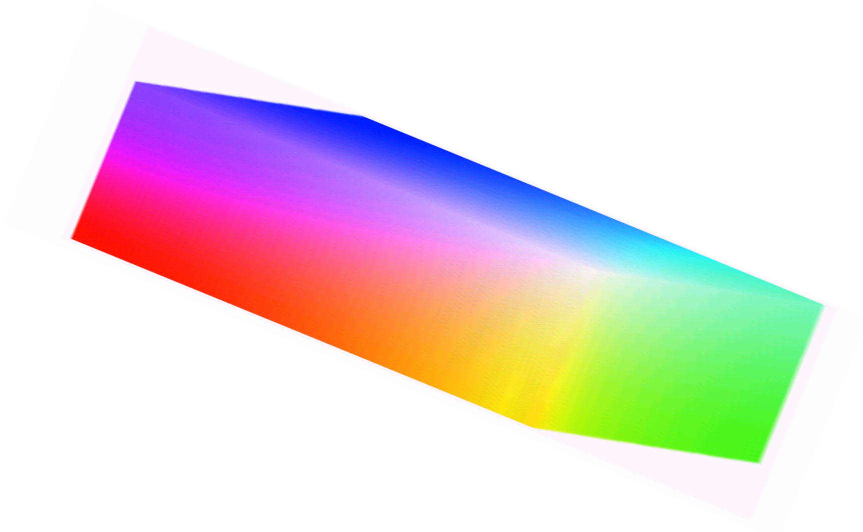




Key algebraic-geometric notions arising from this duality:

3. Strong Regularity

(= Jeřàbek's anchoredness property)

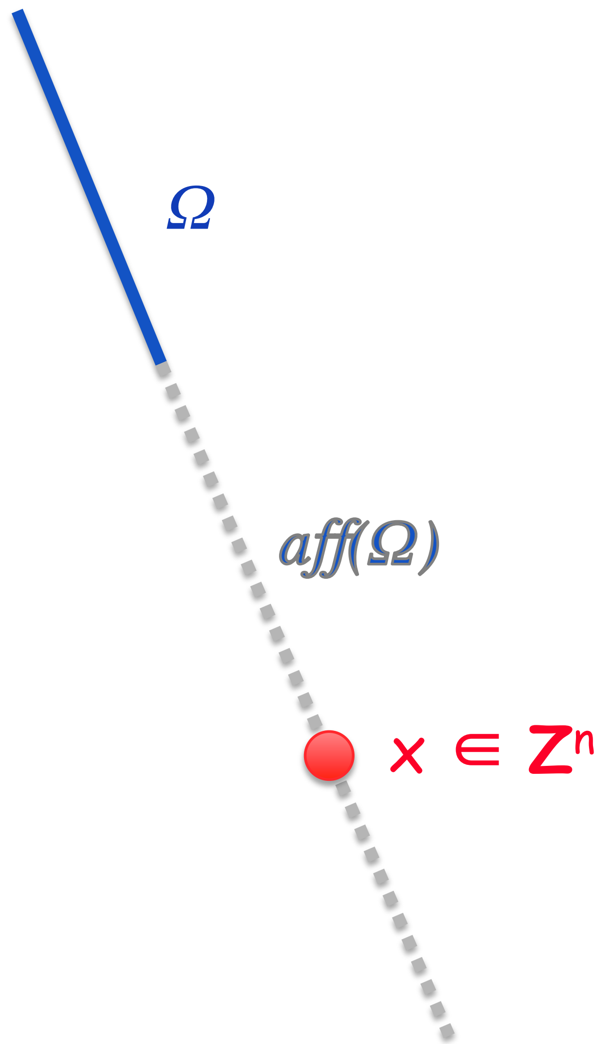


strong regularity

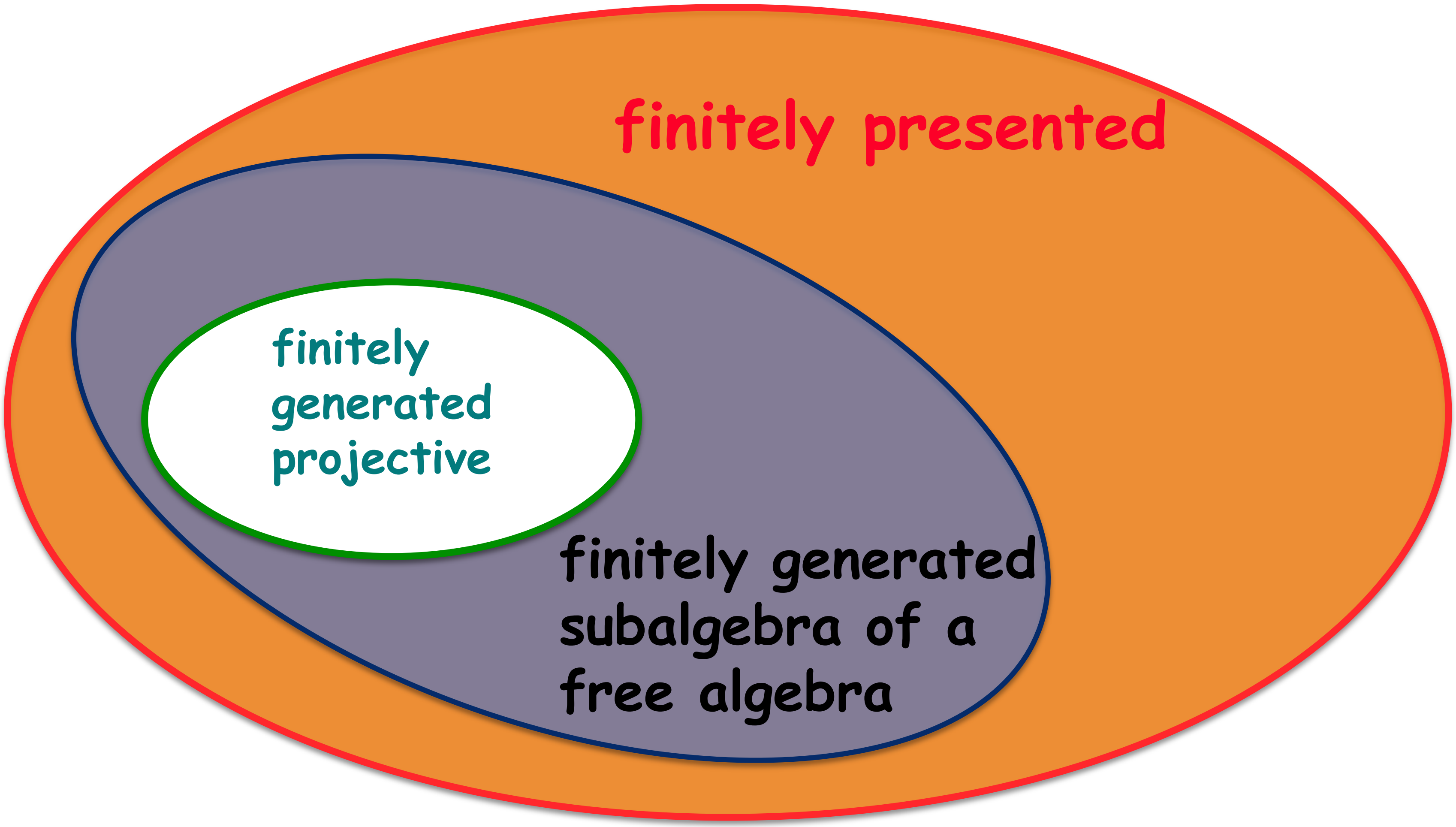
A rational polyhedron P is **strongly regular** if for some (equivalently, for every) regular triangulation Ω of P the affine hull of every maximal simplex of Ω contains an integer point

Equivalently: the denominators of the vertices of every maximal simplex in Ω are relatively prime

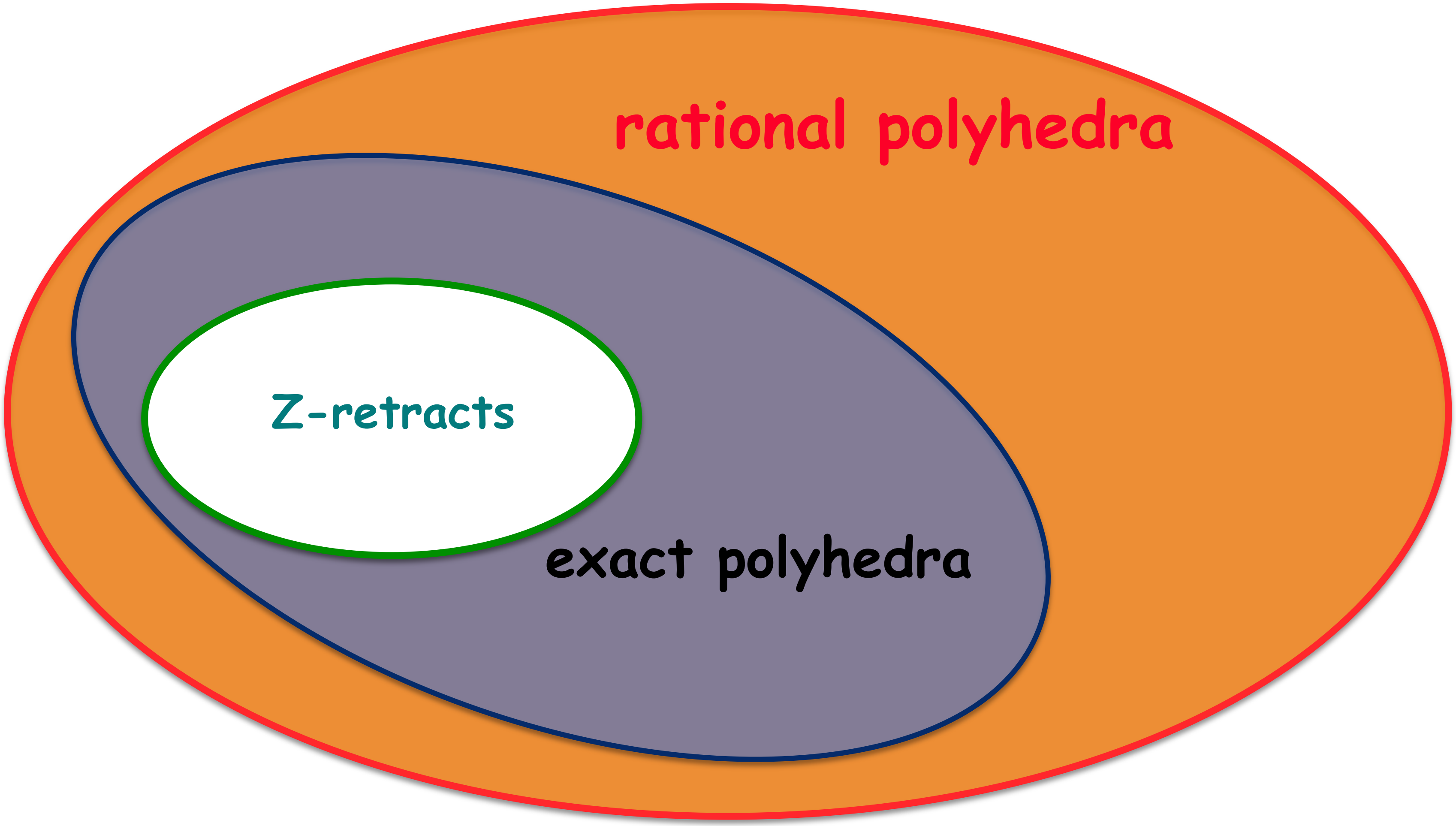
This notion was independently introduced by Jerabek in his analysis of admissibility in the proof-theory of Lukasiewicz logic



three classes of algebras



three classes of polyhedra

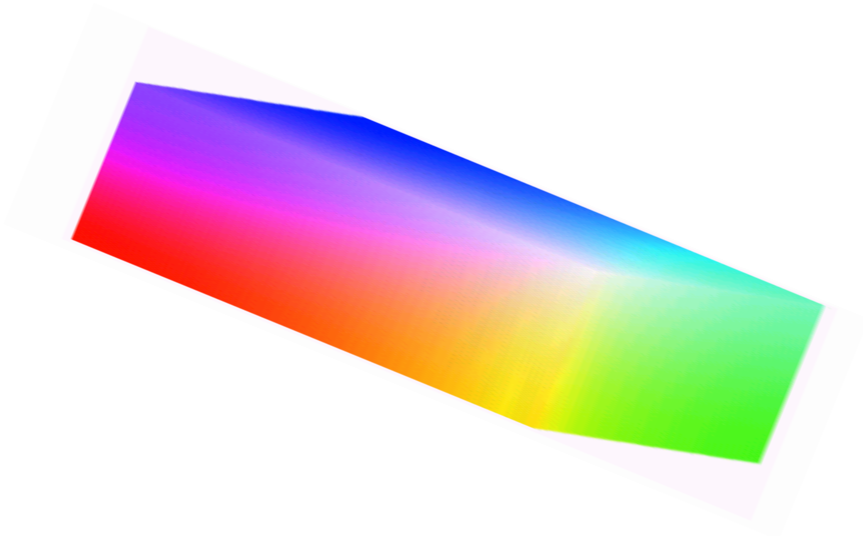


a rational polyhedron in $[0,1]^n$ is **exact** if it contains a vertex of $[0,1]^n$, is strongly regular and connected (L.M.Cabrer, Forum Math. 2015)

algebra	geometry+arithmetic
A is finitely presented	$A=\mathcal{M}(P)$, P a rational polyhedron
homomorphism	**Z**-map
isomorphism	**Z**-homeomorphism
A is indecomposable	P is connected
A is free n-generated	P is the unit cube $[0,1]^n$
A is n-generated	P lies in $[0,1]^n$
dim(maxspec(A)) = d	dim(P) = d
A is a finitely generated subalgebra of a free algebra	P is exact (connected, with a boolean point, strongly regular)
$A = \mathcal{M}(P)$ is projective	**how does P look like ?**

Key algebraic-geometric notions arising from this duality:

4. Z-retracts



Z-retract = dual of finitely generated projective

- As we have seen, *every n-generated projective algebra A is finitely presented, whence by duality we can write* **A=$\mathcal{M}$(P)** *for some polyhedron P lying in the n-cube* $[0,1]^n$.

- DEFINITION *P* is said to be a ***Z-retract (of the n-cube)*** if there is a **Z**-map $\mu: [0,1]^n \rightarrow P$ such that, letting $j : P \rightarrow [0,1]^n$ be inclusion map, the composition $\mu \circ j$ is the identity map on *P*.

-

- COROLLARY *A=$\mathcal{M}$(P) is projective iff P is a **Z**-retract.*

a first property of Z-retracts

If P is a Z-retract then P contains a vertex of the cube.

Proof. By definition, there is a piecewise linear retraction $\mu\colon [0,1]^n \to P$, each linear piece having integer coefficients. Thus μ sends each rational x of $[0,1]^n$ into a rational point y of P whose denominator divides the denominator of x. In particular, every vertex of $[0,1]^n$ is sent into some vertex of $[0,1]^n$. **QED**

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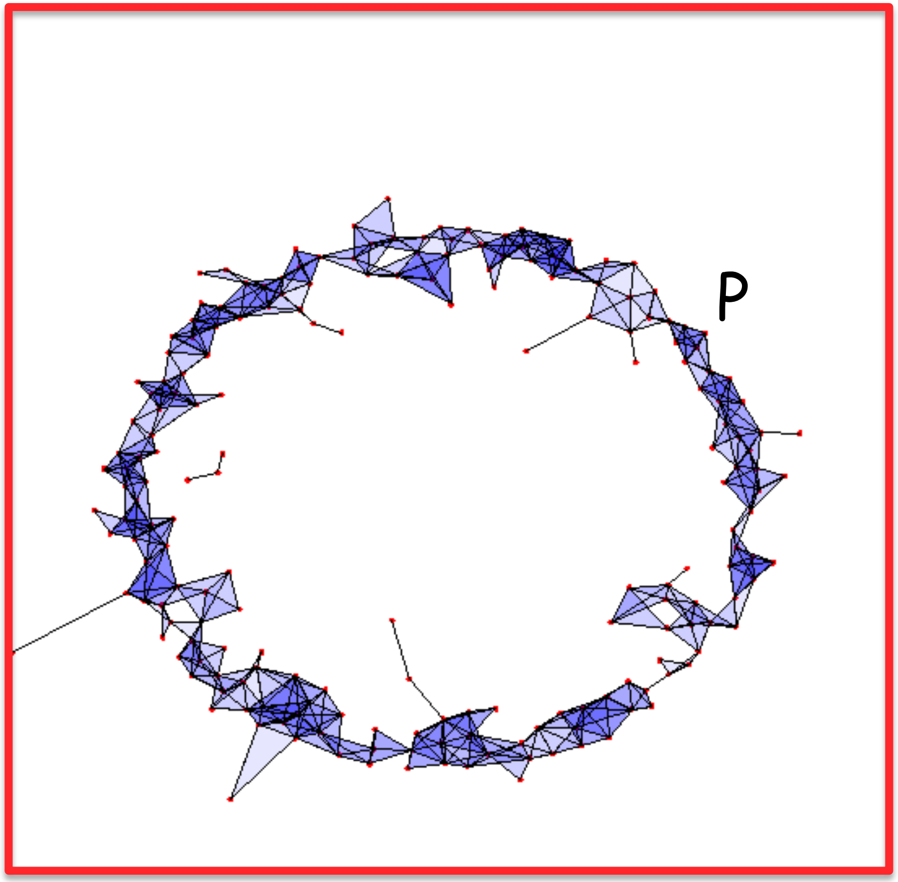

$\mathcal{M}$(P) is not projective

a second property of Z-retracts

THEOREM

(L.Cabrer, D.M., Communications in Contemporary Math. 2012, op.cit.)

If P is a ***Z****-retract, then P is strongly regular.*

a **second** property of Z-retracts

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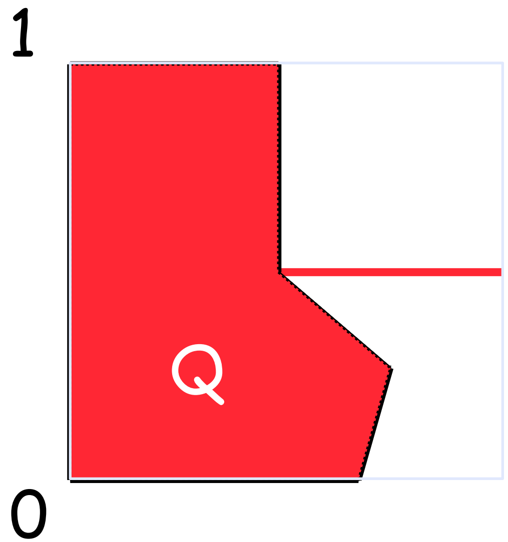

the polyhedron Q is not a Z-retract: the red segment is regular, maximal, but the gcd of the denominator of its vertices is 2

$\mathcal{M}$(Q) is not projective

a third property of Z-retracts

OBSERVATION *If P is a* ***Z****-retract, then, a fortiori, P is a* ***retract*** *of some n-cube.*

THEOREM. *For any polyhedron P in* $[0,1]^n$ *the following conditions are equivalent:*

(a) P is a retract of $[0,1]^n$

(b) P is connected and all homotopy groups $\pi_i(P)$ *are trivial*

(c) P is contractible.

Proof. (a)→(b) by the functorial properties of the homotopy groups π_i . The implications (b)→(a) and (b)→(c) follow from **Whitehead theorem** in algebraic topology. (c)→(b) is a routine exercise in algebraic topology. **QED**

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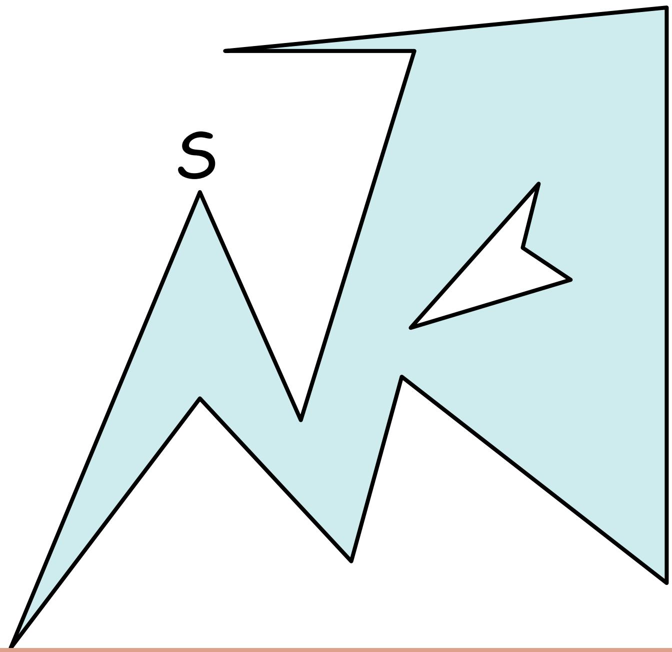

$\mathcal{M}$(S) is not projective

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the geometry of projective MV-algebras

THEOREM (L. CABRER, D.M., Comm. Contemporary Math. 2012)

If A is a finitely generated projective MV-algebra or a unital abelian l-group, writing without loss of generality $A=\mathcal{M}(P)$ *for some rational polyhedron P in* $[0,1]^n$ *it follows that*

(i) P contains some vertex of $[0,1]^n$,

(ii) P is contractible, and

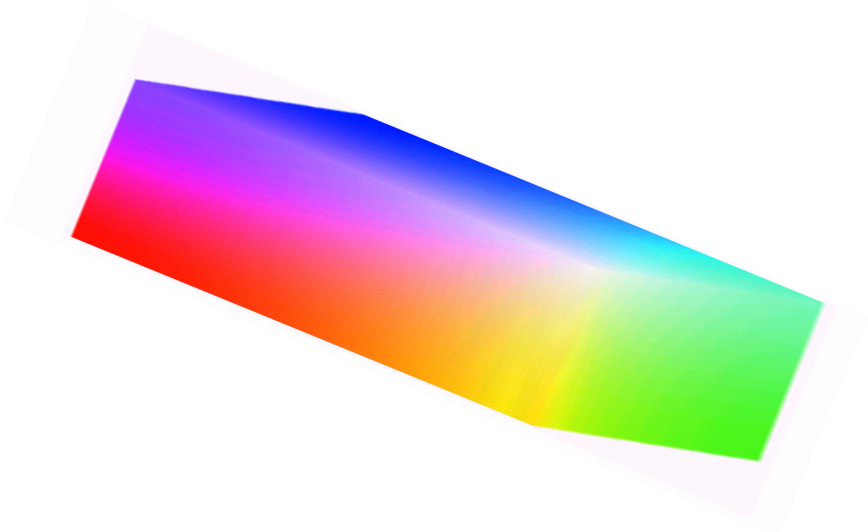
(iii) P is strongly regular.

For the **converse of this theorem** see L.M.CABRER's paper in arXiv 1405.7118 (a tour de force in algebraic topology)

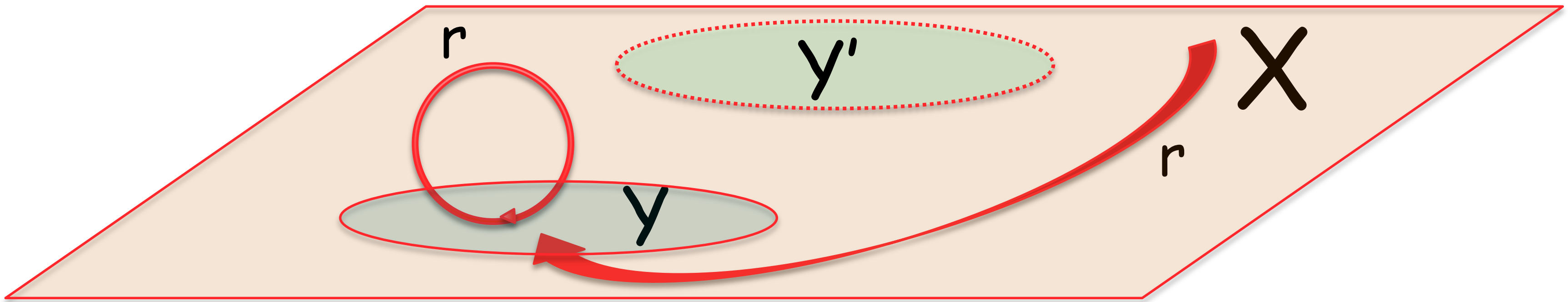
This completes the geometric algebraic topological excursion needed to characterize finitely generated projective MV-algebras and unital *l*-groups

Key algebraic-geometric notions arising from this duality:

5. The projectivity index



Idempotent endomorphisms



We all know what a retraction r : X→Y is. The map r acts identically on its range, $r^2=r$. *We are seldom interested in the behavior of r over the domain* X\Y. For instance, there might be a region Y'≠Y where r acts isomorphically onto Y.

And yet, the behavior of r outside its range may be decisive for the construction of new invariants for projective objects

Think of your favorite (quasi)variety
Let F be the free n-generator Q-algebra
Let A be a retract of F

- Thus there is at least one *retraction* $r = r^2$ of F onto A
- **Problem 1.** Under which conditions the number of retractions of F onto A is finite ?
- **Problem 2.** Give a sequence A_i of retracts of of F such that the number of retractions of F onto A_i is finite and $> i$

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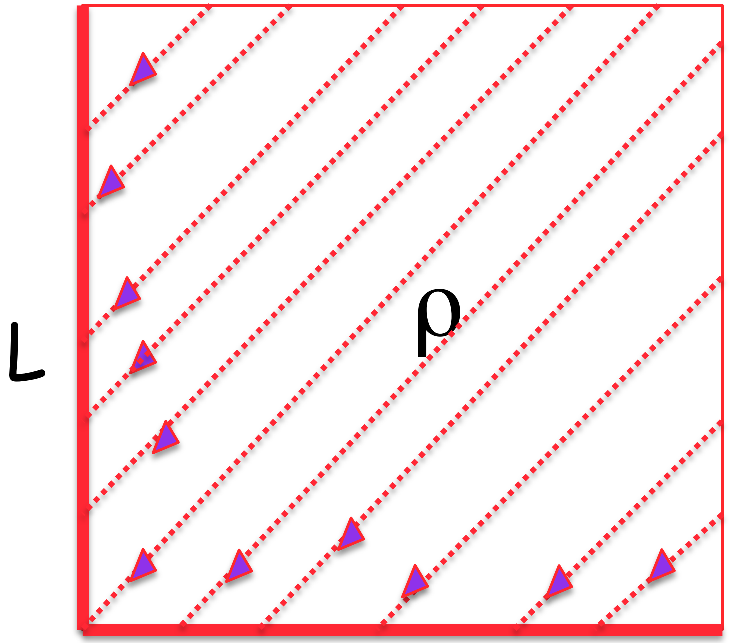
Answer to Problem 1 for MV-algebras (L.Cabrer, D.M. 2015):
The number of retractions onto A is finite iff the maximal space R of $A = \mathcal{M}(R)$ is a closed domain.
(i.e., R is equal to the closure of its interior in $[0,1]^n$)

Example of a retract A of $FREE_2$ such that infinitely many retractions exist of $FREE_2$ onto A



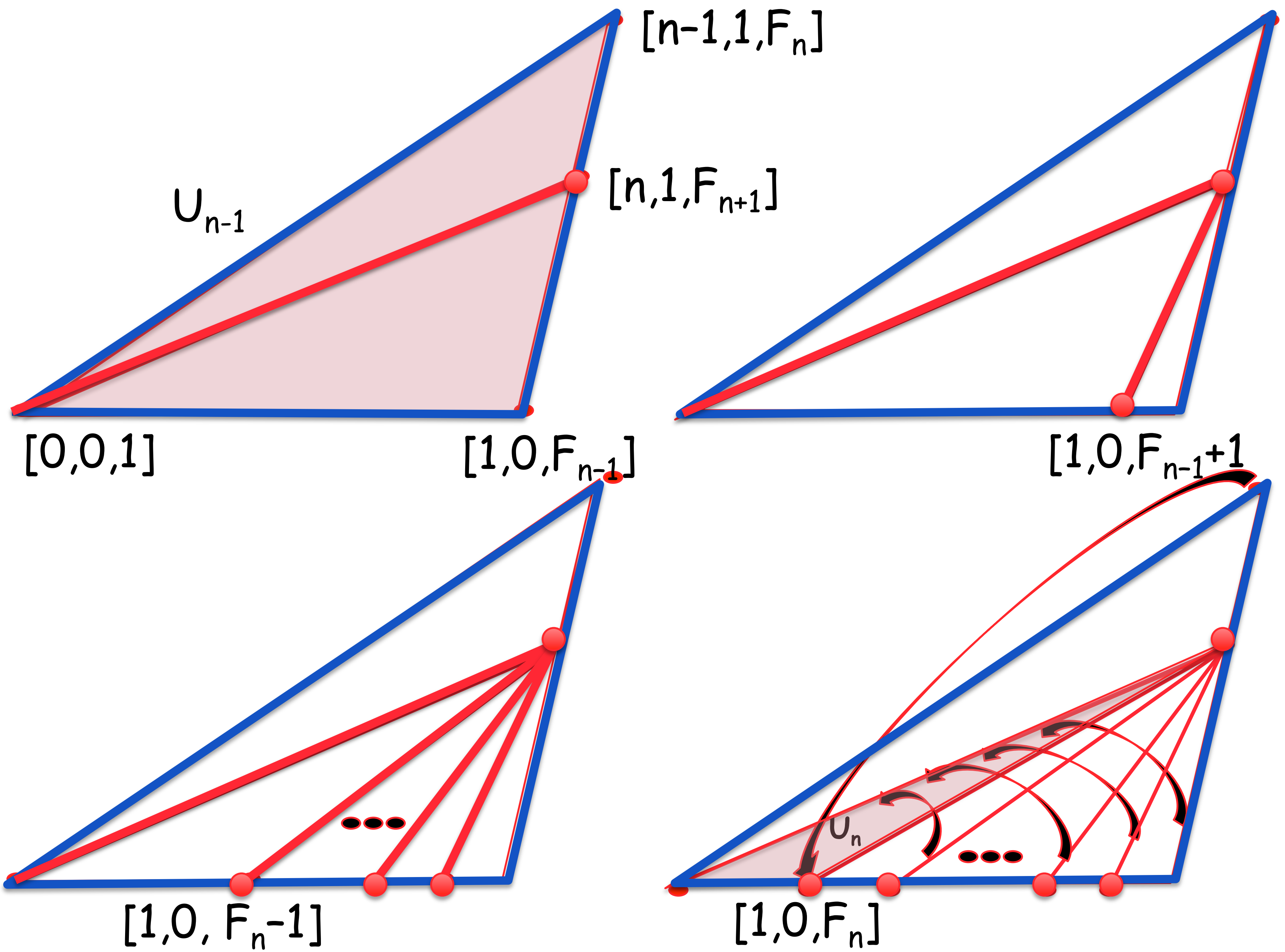
A = $\mathscr{M}$(*L*) = the MV-algebra of all restrictions to *L* of the McNaughton functions of the free 2-generator MV-algebra $\mathscr{M}([0,1]^2)$. *A* dually corresponds to the **Z**-retraction ρ of the unit square onto *L*

Problem 2. For every i=1,2,..., construct a retract A_i of F such that there are > i (but finitely many) retractions of $FREE_2$ onto A_i

Answer (n=2):

$$A_i = \mathcal{M}(U_i)$$

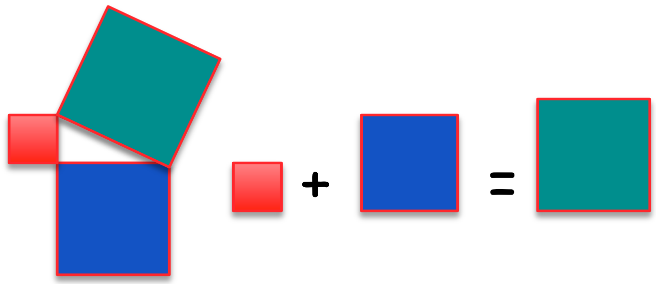
where U_i , U_{i+1} are the coloured triangles in the next picture, F_i= the ith **Fibonacci** number, and the red points are given by **Farey** mediants. Points are specified in **homogeneous coordinates**


[n−1,1,F_n]
[n,1,F_{n+1}]
U_{n-1}
[0,0,1]
[1,0,F_{n-1}]
[1,0,F_{n-1}+1]
[1,0, F_n−1]
U_n
[1,0,F_n]

closing a circle of ideas

We have just seen three functors in action, between MV-algebras, rational polyhedra and unital *l*-groups. Their existence relies upon deep theorems in algebraic topology, polyhedral geometry, algebra, and many-valued logic.

$$(x \oplus y) \oplus z = x \oplus (y \oplus z)$$
$$x \oplus y = y \oplus x$$
$$x \oplus 0 = x$$
$$\neg\neg x = x$$
$$x \oplus \neg 0 = \neg 0$$
$$\neg(y \oplus \neg x) \oplus y = \neg(x \oplus \neg y) \oplus x$$



THANK YOU

THANK YOU

MV and *l*-groups: of course
MV and Riesz spaces: Cabrer, Di Nola. Lapenta, Leustean, Pedrini
MV and Differential geometry: Busaniche, Cabrer, D.M.
MV and Semirings, tropical mathematics: Belluce, Di Nola, Ferraioli, Russo
MV and Probability: Flaminio, Keimel, Montagna†, Rieçan
MV and Games: Kroupa, Teheux
MV and Multisets: Cignoli, Marra, Nganou
MV and Semantics of Lukasiewicz logic: Picardi, D.M.
MV and Proof-theory of Lukasiewicz logic: Cabrer, Ciabattoni, Jeràbek, Metcalfe
MV and Modal logic, Belief: Flaminio, Godo, Kroupa, Teheux
MV and Quantum structures: Dvureçenskij, Pulmannovà
MV and AF C*-algebras: Lawson, Scott, D.M.
MV and Discrete Dynamical Systems: Cabrer, D.M.
MV and Categories, Morita equivalence, coordinatization, duality, sheafs:
Caramello, Gehrke, Lawson, Marra, Russo, Scott, Spada