Frames, topologies, and duality theory

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Lecture 3

Recap

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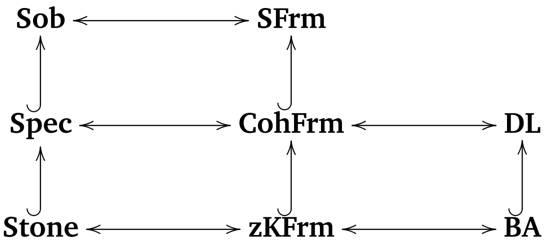

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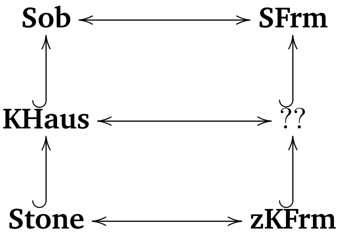

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$a \prec b$ iff $a^* \vee b = 1$.

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Proof sketch:

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$$
\begin{array}{ccc}
\mathbf{Sob} & \longleftrightarrow & \mathbf{SFrm} \\
\uparrow & & \uparrow \\
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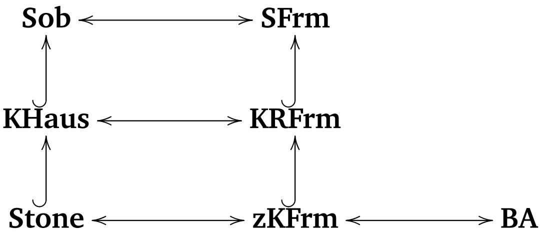

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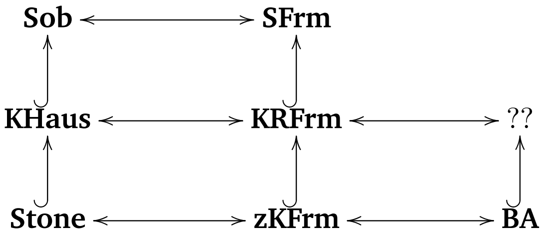

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$\mathfrak{B}(L)$ is called the Booleanization of L.

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If $h : L \to M$ is a frame homomorphism, then h may not send regular elements to regular elements. So we define $\mathfrak{B}(h) : \mathfrak{B}(L) \to \mathfrak{B}(M)$ by $\mathfrak{B}(h)(a) = h(a)^{**}$.

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To see (4), $\mathfrak{B}(h)(a) = (\bigvee\{h(b) : b \prec a\})^{**} \leqslant (\bigvee\{\mathfrak{B}(h)(b) : b \prec a\})^{**} = \bigvee_{\mathfrak{B}(M)}\{\mathfrak{B}(h)(b) : b \prec a\} \leqslant \mathfrak{B}(h)(a)$.

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De Vries duality

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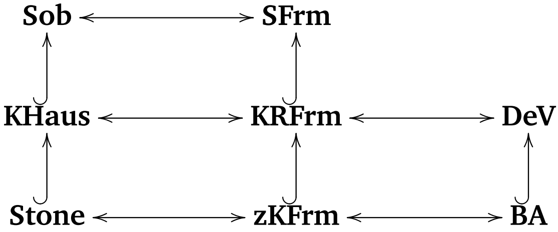

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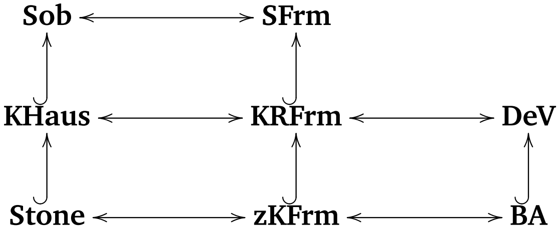

Theorem (de Vries): KHaus is dually equivalent to **DeV**.

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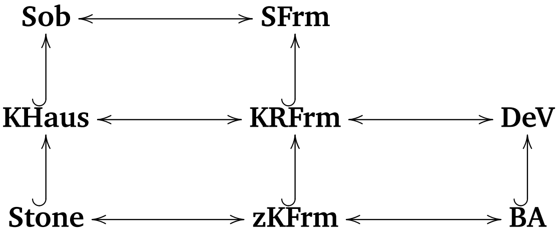

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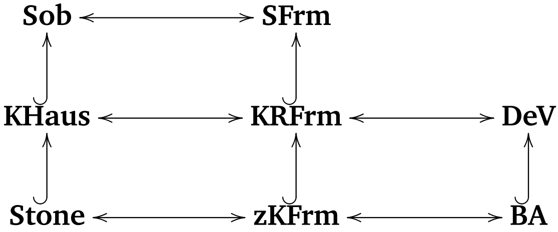

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Note: The de Vries functor $\mathcal{RO}$: **KHaus** $\to$ **DeV** is the functor of taking regular open sets.

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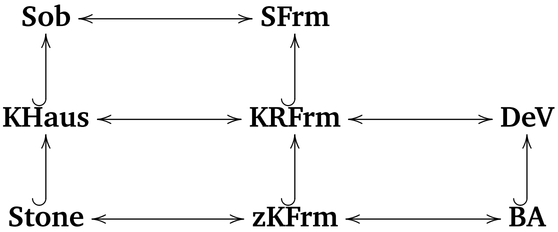

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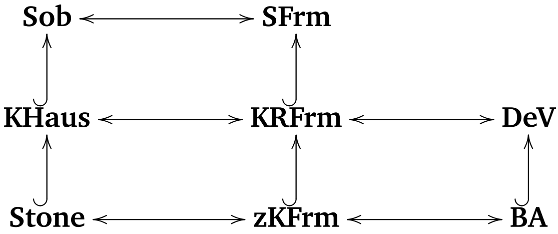

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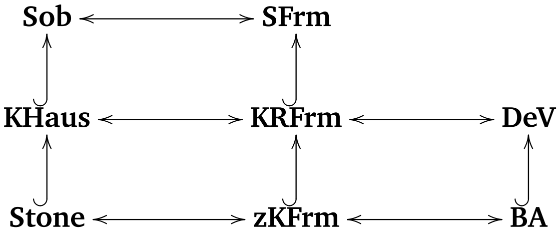

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The other de Vries functor $\mathcal{E}$: **DeV** $\rightarrow$ **KHasu** is the functor of taking ends (maximal round filters). Since ends of a de Vries algebra correspond to the points of its round ideals, it is exactly the composition $pt \circ \mathfrak{R}$.

End of Lecture 3