Frames, topologies, and duality theory

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Lecture 2

Recap

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The contravariant adjunction Ω : **Top** $\to$ **Frm**, pt : **Frm** $\to$ **Top** restricts to the dual equivalence between the full subcategory **Sob** of **Top** consisting of sober spaces and the full subcategory **SFrm** of **Frm** consisting of spatial frames.

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Recall that a space is sober if each irreducible closed set has a unique generic point, and a frame is spatial if its different elements can be separated by points.

More about sober spaces

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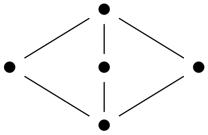

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But in general there could exist irreducible closed sets that are not closures of points.

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But there exist T_1-spaces that are not sober.

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Then $p \in pt\, L$, $p(a) = 1$, and $p(b) = 0$. Thus, L is spatial.

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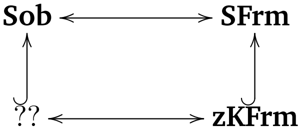

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Thus, **Stone** is a full subcategory of **Sob**.

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Proof sketch: Clearly if X is a compact space, then $\Omega(X)$ is a compact frame. An open set U is complemented iff U is clopen. Therefore, $Z(\Omega\, X)$ is exactly the clopens of X. Thus, if X is a zero-dimensional space, then $\Omega(X)$ is a zero-dimensional frame. So $X \in \mathbf{Stone}$ implies $\Omega(X) \in \mathbf{zKFrm}$.

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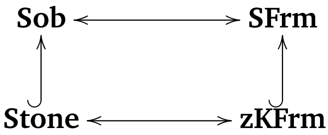

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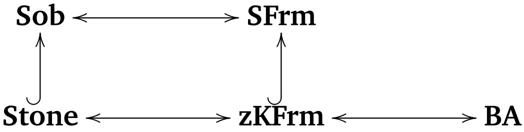

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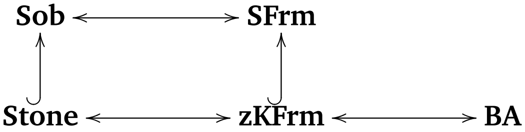

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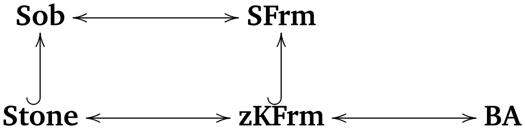

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Note: Stone's functors are nothing more but the compositions $Z \circ \Omega$ and $pt \circ \mathfrak{I}$!

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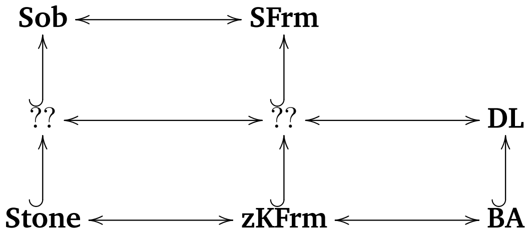

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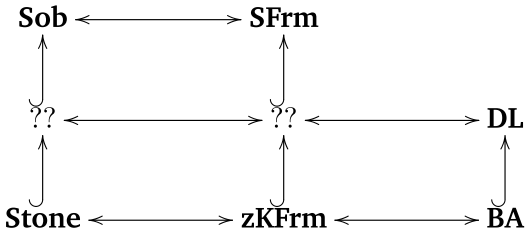

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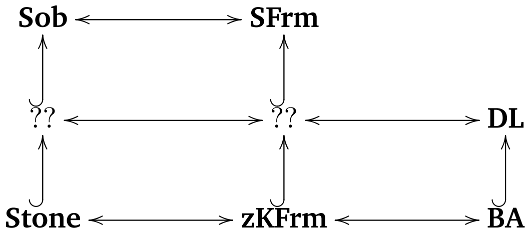

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The idea is the same as in the case of **BA**; that is, for a bounded distributive lattice D, we look at the poset $\mathfrak{I}(D)$ of ideals of D.

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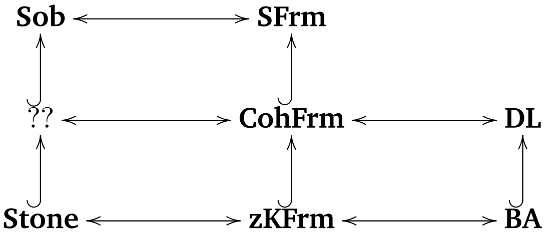

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Stone duality for distributive lattices

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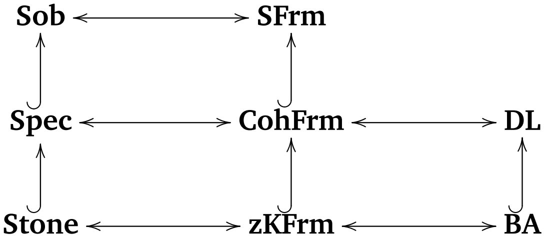

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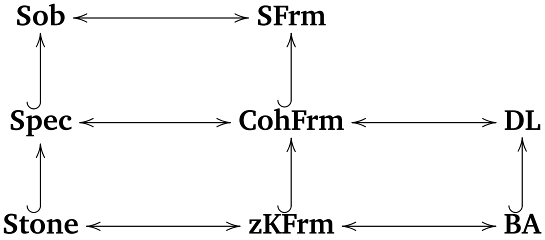

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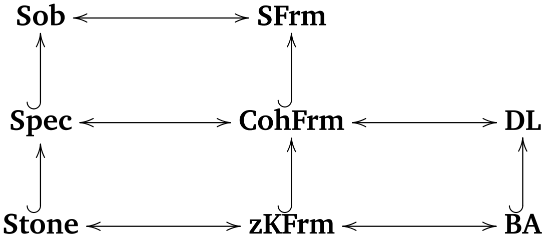

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Priestley duality for distributive lattices

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Definition: A Priestley space is a compact ordered space satisfying the Priestley separation axiom.

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Note: Priestley functors are the compositions.

End of Lecture 2