Frames, topologies, and duality theory

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Lecture 1

Introduction

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(3) Algebraic Topology (homotopy groups are used to classify topological spaces).

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(5) Functional Analysis (commutative C^*-algebras are the algebras of continuous complex-valued functions on compact Hausdorff spaces).

(6) Logic (the study of Lindenbaum algebras of different logical systems through their spectra provides connection between logic, algebra, lattice theory, and topology).

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It is easy to verify that topological spaces and continuous maps between them form a category, which we denote by **Top**.

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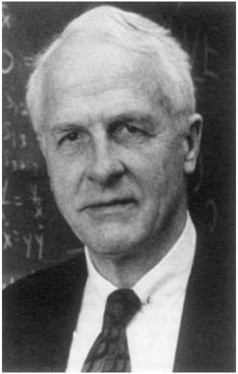

Garrett Birkhoff (1911-1996)

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The partial order on L is given by $a \leqslant b$ iff $a = a \wedge b$ (equivalently, $a \leqslant b$ iff $a \vee b = b$), and $a \vee b$ is the supremum and $a \wedge b$ is the infimum of $\{a, b\}$.

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Bounded lattices have two additional constants $0, 1$ satisfying

$$0 \wedge a = 0 \text{ and } a \vee 1 = 1.$$

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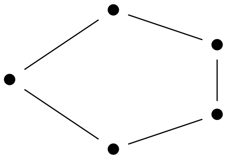
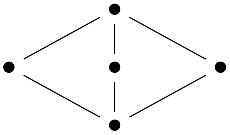


N_5



M_3

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Unlike the finite distributive laws, JID and MID are not equivalent!

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It is easy to see that frames and frame homomorphisms between them form a category, which we denote **Frm**.

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But how do we go back? In other words, how do we associate a topological space with a frame?

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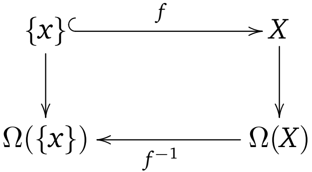

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Definition: A point of a frame L is a frame homomorphism $p : L \to \mathbf{2}$.

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\end{aligned}
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$$
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Thus, points $\Leftrightarrow$ completely prime filters $\Leftrightarrow$ meet prime elements

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It follows that $O : L \to \tau$ is a frame homomorphism, and hence τ is a topology on $pt(L)$.

Frame homomorphisms and continuous maps

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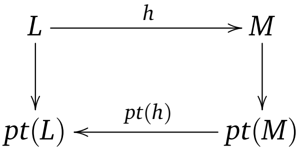

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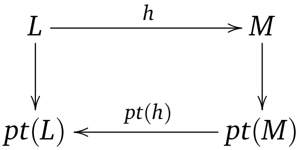

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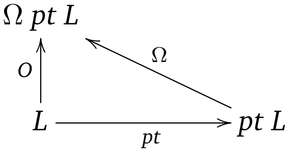

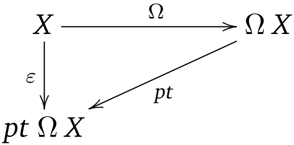

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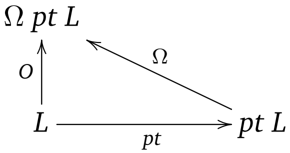

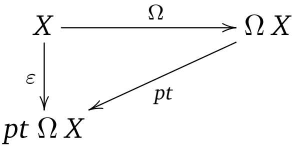

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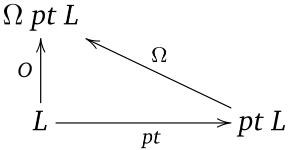

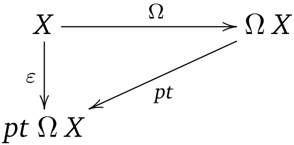

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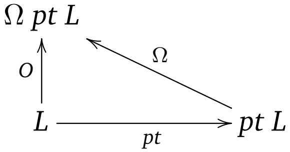

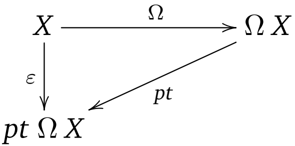

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Note: ε is continuous because $\varepsilon^{-1}O(U) = U$ for all $U \in \Omega X$.

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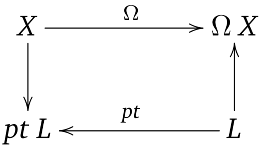

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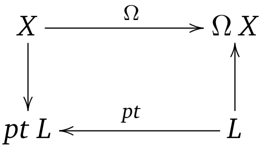

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How can we utilize this framework? In particular, how can it be used in Logic?

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How are Boolean algebras related to frames?

From Boolean algebras to frames

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Proof: It is sufficient to observe that $\{I_\alpha\} \subseteq \mathfrak{I}(B)$ implies $\bigcap_\alpha I_\alpha \in \mathfrak{I}(B)$.

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Therefore, principal ideals are complemented in $\mathfrak{I}(B)$.

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End of Lecture 1