A uniform continuity principle for the Baire space and a corresponding bar induction

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Constructive mathematics and uniform continuity principle

We work in the constructive mathematics started by E. Bishop in his book “Foundation of Constructive Analysis”.

- Based on intuitionistic logic
- Compatible with classical mathematics (CLASS), but also
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- **UC** is true in CLASS and INT
- **UC** contradicts with RUSS

Uniform continuity principle and Fan theorem

Fan theorem "Every bar has a uniform upper bound."

For any $P \subseteq \mathbf{2}^*$,

$$(\forall \beta \in \mathbf{2}^{\mathbb{N}})(\exists n \in \mathbb{N})P(\overline{\beta}n) \implies (\exists N \in \mathbb{N})(\forall \beta \in 2^{\mathbb{N}})P(\overline{\beta}N),$$

where $\overline{\beta}n \stackrel{\text{def}}{=} \langle \beta(0), \ldots, \beta(n-1) \rangle$.

Theorem (J. Berger, 2006)

$\mathbf{UC}$ *is equivalent to* $\mathbf{c\text{–}FT}$ *(continuous Fan theorem).*

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Question. Does the equivalence admit a natural generalisation to the Baire space $\mathbb{N}^{\mathbb{N}}$? Is there any uniform continuity principle for the Baire space which corresponds to some version of Bar induction?

Outline

1. $\mathbf{UC_B}$: Uniform continuity principle for the Baire space
2. $\mathbf{c\text{–}BI}$: Continuous Bar induction
3. Equivalence $\mathbf{UC_B} \iff \mathbf{c\text{–}BI}$
4. Relation to $\mathbf{UC}$ and $\mathbf{c\text{–}FT}$

Neighbourhood functions $\alpha\colon \mathrm{N}^* \to \mathrm{N}$

The class $K \subseteq \mathbb{N}^* \to \mathbb{N}$ of **neighbourhood functions** is inductively generated by the following clauses:

$$\frac{n \in \mathbb{N}}{\lambda a.n+1 \in K} \qquad \frac{\alpha(\langle\rangle) = 0 \quad (\forall n \in \mathbb{N})\lambda a.\alpha(\langle n\rangle * a) \in K}{\alpha \in K}$$

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Remark. A neighbourhood function $\alpha \in K$ can be identified with a well-founded tree labelled by elements of $\mathbb{N}$.

1. $\lambda a.n+1$ corresponds to a single node tree $\{(\langle\rangle, n+1)\}$ labelled by $n+1$.
2. if $\alpha(\langle\rangle) = 0$ and for each $n \in \mathbb{N}$, $\lambda a.\alpha(\langle n\rangle * a)$ corresponds to a labelled tree T_n, then α corresponds to a tree $T = \{(\langle\rangle, 0)\} \cup \{(\langle n\rangle * a, L) \mid n \in \mathbb{N}, (a, L) \in T_n\}$.

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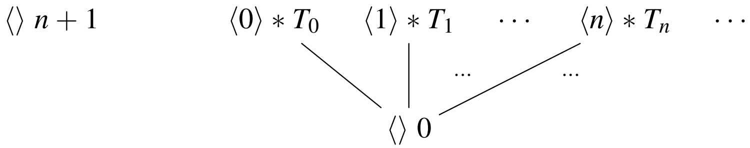

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The leaves of the tree corresponding to neighbourhood function $\alpha \in K$ determines a bar

$$P_\alpha = \left\{a \in \mathbb{N}^* \mid \alpha(a) > 0 \;\&\; \left(\forall a' \prec a\right) \alpha(a') = 0\right\},$$

that is $\left(\forall \beta \in \mathbb{N}^{\mathbb{N}}\right) (\exists k \in \mathbb{N})\, \overline{\beta}k \in P_\alpha$.

A neighbourhood function $\alpha \in K$ determines a (unique) continuous function $f_\alpha \colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ such that for each $\beta \in \mathbb{N}^{\mathbb{N}}$

$$f_\alpha(\beta) = \alpha(\overline{\beta}k) - 1$$

where $k \in \mathbb{N}$ is such that $\overline{\beta}k \in P_\alpha$.

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Definition

A function $f\colon\ \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is **realizable** if $f = f_\alpha$ for some $\alpha \in K$.

$\mathbf{UC_B}$ Every point-wise continuous function $f\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is realizable.

Proposition

A function $f\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is realizable iff there exists $\alpha \in K$ such that

$$(\forall a \in P_\alpha)\,(\forall \beta, \gamma \in a)\; f(\beta) = f(\gamma),$$

where $\beta \in a \overset{\text{def}}{\iff} \overline{\beta}|a| = a$, i.e. f is uniformly continuous with respect to the covering uniformity $\{P_\alpha \mid \alpha \in K\}$ on $\mathbb{N}^{\mathbb{N}}$.

c–BI : continuous Bar induction.

- $P \subseteq \mathbb{N}^*$ is a **bar** if $\left(\forall \beta \in \mathbb{N}^{\mathbb{N}}\right)(\exists n \in \mathbb{N})\, P(\overline{\beta} n)$.
- A bar P is a **c–bar** if there is $\delta\colon\ \mathbb{N}^* \to \mathbb{N}$ such that $P(a) \leftrightarrow (\forall b \in \mathbb{N}^*)\, \delta(a) = \delta(a * b)$ for all $a \in \mathbb{N}^*$.
- $Q \subseteq \mathbb{N}^*$ is **inductive** if $(\forall n \in \mathbb{N})\, Q(a * \langle n \rangle) \to Q(a)$ for all $a \in \mathbb{N}^*$.

c–BI If P is a c–bar and Q is an inductive subset such that $P \subseteq Q$, then $Q(\langle\rangle)$.

Equivalence

Theorem. $\mathbf{UC_B}$ and $\mathbf{c\text{–}BI}$ are equivalent.

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Proof. ($\mathbf{c\text{–}BI} \implies \mathbf{UC_B}$) Let $f\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ be a continuous map. Write $f(a) \stackrel{\text{def}}{=} f(a * 0^{\omega})$ for each $a \in \mathbb{N}^*$. Then

$$\left(\forall \beta \in \mathbb{N}^{\mathbb{N}}\right) (\exists n \in \mathbb{N}) (\forall b \in \mathbb{N}^*) f(\overline{\beta}n) = f(\overline{\beta}n * b).$$

Define $\delta\colon \mathbb{N}^* \to \mathbb{N}$ by $\delta(a) \stackrel{\text{def}}{=} f(a)$, and put

$$P(a) \stackrel{\text{def}}{\iff} (\forall b \in \mathbb{N}^*)\, \delta(a) = \delta(a * b).$$

Then P is a c–bar. Define

$$Q(a) \stackrel{\text{def}}{\iff} (\exists \alpha \in K) (\forall b \in \mathbb{N}^*)\, \alpha(b) > 0 \to P(a{*}b) \;\&\; \alpha(b) = \delta(a{*}b){+}1.$$

It can be shown that Q is inductive and $P \subseteq Q$. By $\mathbf{c\text{–}BI}$, we get $Q(\langle\rangle)$, so there exists $\alpha \in K$ such that

$$(\forall a \in \mathbb{N}^*)\, \alpha(a) > 0 \implies \alpha(a) = f(a) + 1,$$

which means $f = f_{\alpha}$. □

Lemma 1. *For any $\alpha \in K$ and $Q \subseteq \mathbb{N}^*$ which is inductive*

$$P_\alpha \subseteq Q \implies Q(\langle\rangle).$$

Lemma 2. *For any $\alpha \in K$, there is $\alpha' \in K$ such that*

$$(\forall a \in \mathbb{N}^*)\, \alpha'(a) > 0 \implies \alpha(a) = \alpha'(a) \;\&\; \alpha'(a) < |a|.$$

Proof of $\mathbf{UC_B} \implies$ c–BI. Let $P \subseteq \mathbb{N}^*$ be a c–bar and $Q \subseteq \mathbb{N}^*$ be an inductive subset such that $P \subseteq Q$. Then, there exists $\delta\colon\ \mathbb{N}^* \to \mathbb{N}$ such that $P(a) \iff (\forall b \in \mathbb{N}^*)\, \delta(a) = \delta(a * b)$. We must show $Q(\langle\rangle)$.
By Lemma 1, it suffices to find $\alpha \in K$ such that $P_\alpha \subseteq Q$.
Let $\beta \in \mathbb{N}^{\mathbb{N}}$. Since P is a bar, there exists $n \in \mathbb{N}$ such that $P(\overline{\beta}n)$. Put

$$D_\beta \overset{\text{def}}{=} \left\{m \in \mathbb{N} \mid \delta(\overline{\beta}m) \neq \delta(\overline{\beta}n)\right\} \cup \{1\},$$

and define a continuous $f\colon\ \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ by

$$f(\beta) \overset{\text{def}}{=} \max D_\beta.$$

By $\mathbf{UC_B}$ and Lemma 2, there exists $\alpha \in K$ such that $f = f_\alpha$ and $\alpha(a) > 0 \to |a| > \alpha(a)$. Let $a \in P_\alpha$. Then $\alpha(a) > 0$, so

$$|a| > \alpha(a) = f_\alpha(a) + 1 = \max D_{a*0^\omega} + 1.$$

Then, $(\forall b \in \mathbb{N}^*)\, \delta(a) = \delta(a * b)$, i.e. $P(a)$. Hence $Q(a)$. $\square$

Relation to UC and c–FT

The class $K \subseteq \mathbb{N}^*$ of neighbourhood functions is inductively generated by the following clauses:

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Relation to UC and $\mathbf{c\text{–}FT}$

The class $K_{\mathbf{C}} \subseteq \mathbf{2}^*$ of neighbourhood functions is inductively generated by the following clauses:

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Proposition. *A function $f : \mathbf{2}^{\mathbb{N}} \to \mathbb{N}$ is uniformly continuous iff f is realizable by some $\alpha \in K_{\mathbf{C}}$.*

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c–FT' If P is a c–bar and Q is an inductive subset such that $P \subseteq Q$, then $Q(\langle\rangle)$.

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Proposition. **c–FT'** *and* **c–FT** *are equivalent.*

Further works and references

Further work

- Is an analogy between $\mathbf{UC_B} \iff \mathbf{c\text{–}BI}$ and $\mathbf{UC} \iff \mathbf{c\text{–}FT}$ can be formulated in more mathematical way?
- $\mathbf{BI_M} \implies \mathbf{c\text{–}BI} \implies \mathbf{BI_D}$. Are these implications strict?

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