Constructing representations of ordered monoids

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Cayley representations

A ... monoid, B ... A-module,
$A \to \mathrm{End}(B)$ $(= B \multimap B)$, $a \mapsto a-$.

We are interested in ordered monoids, idempotent semirings, quantales, etc.

$$\begin{array}{ccc} A & \longrightarrow & B \multimap B \\ \downarrow & & \downarrow \\ \hat{A} & \longrightarrow & \hat{B} \multimap \hat{B} \end{array}$$

(1) Category background.

(2) Matrix modules over quantales.

Categories

- Set — sets, mappings, semigroups, monoids,
- Pos — preordered sets, isotone mappings, po-semigroups, po-monoids,
- Slat — join semilattices, semilattice morphisms, idempotent semirings, unital id. s.,
- Sup — complete lattices, mappings preserving suprema, quantales, unital quantales.

Finer hierarchy: near semilattices, prequantales, etc.

Inclusions and "free" reflections

$$\mathrm{Set} \underset{U_1}{\overset{F_1}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{Pos} \underset{U_2}{\overset{F_2}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{Slat} \underset{U_3}{\overset{F_3}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{Sup}$$

The Fs are standard completions:

- F_1: antichain,
- F_2: finitely generated down-sets,
- F_3: lattice ideals.

Tensor product

All the categories are *symmetric closed monoidal*:
$(A \otimes B) \multimap C \cong A \multimap (B \multimap C)$.

F_i are *strict monoidal*: $F(A \otimes B) \cong FA \otimes FB, F1 \cong 1$.

$A \otimes B$ is "created" as a colimit:

$$
\begin{array}{ccccc}
\coprod_{UB} UA & \longrightarrow & UA \times UB & \longleftarrow & \coprod_{UA} UB \\
\downarrow & & & & \downarrow \\
F(\coprod UA) \cong \coprod FUA & & \Big\downarrow & & \coprod FUB \cong F(\coprod UB) \\
\downarrow & & & & \downarrow \\
\coprod A & \longrightarrow & A \otimes B & \longleftarrow & \coprod B
\end{array}
$$

Semigroup, monoid, and module objects

- Semigroup: $A \otimes A \to A$,
- monoid: $1 \to A$,
- module: $A \otimes B \to B$,
- representation: $A \to B \multimap B$.

$B \multimap B$ is always a monoid object. If B is a (unital) A-module, then the induced morphism $A \to B \multimap B$ is a semigroup/monoid morphism.

Completions

Semigroups/monoids:

$$\begin{array}{ccc} A \otimes A & \longrightarrow & A \\ \downarrow & & \downarrow \\ FA \otimes FA & \longrightarrow & FA \end{array} \qquad 1 \longrightarrow A \longrightarrow FA$$

Modules and representations:

$$\begin{array}{ccc} A \otimes B & \longrightarrow & B \\ \downarrow & & \downarrow \\ FA \otimes FB & \longrightarrow & FB \end{array} \qquad \begin{array}{ccc} A & \longrightarrow & B \multimap B \\ \downarrow & & \downarrow \\ FA & \longrightarrow & FB \multimap FB \end{array}$$

Unital completion

Semigroup object $\to$ monoid object;
module object $\to$ unital module object.

1 is $\{e\}$ for $\mathrm{Set}, \mathrm{Pos}, \mathrm{Slat}$, $\{0, e\}$ for Sup; $F1 = 1$.

$$\begin{array}{ccc} A & \longrightarrow & A + 1 \\ \downarrow & & \downarrow \\ FA & \longrightarrow & FA + 1 \end{array}$$

Conclusion of the category part

Every semigroup/monoid object of Set / Pos / Slat / Sup can be completed to a unital quantale.

Every module/representation of the object extends to a module/representation of the enveloping quantale.

Matrices over quantales

Q . . . quantale,
$A : I \times J \to Q$ . . . Q-valued matrix of type I, J,
$B : J \times K \to Q$ . . . Q-valued matrix of type J, K,
then

$$C(i,k) = \bigvee_{j \in J} A(i,j)B(j,k)$$

provides a Q-valued matrix of type I, K.

Quantaloid of Q-valued matrices.

It is *residuated*:

$$AB \leq C \Leftrightarrow A \leq B \to C \Leftrightarrow B \leq C \leftarrow A$$

Matrix modules

A ... matrix of type I, J,
Q^I ... all column vectors of type I,
Q_J ... all row vectors of type J,
$xy \leq A$ ... Galois connection between Q^I and Q_J:

$$y \mapsto y \to A \qquad\qquad x \mapsto A \leftarrow x$$

$Q_J \to A = \{y \to A \mid y \in Q_J\}$ with $\bigwedge, \to$ is a left Q-module:
$a \to (y \to A) = (ay) \to A$, etc.

Classification of unital quantale modules

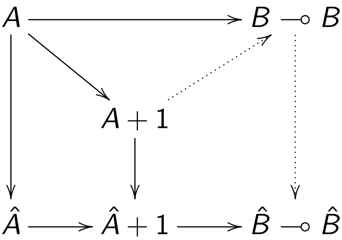
[DK 2002] Every unital left Q-module M is isomorphic to a matrix module $Q_M \to A$ for A of type M, M given by $A(n, m) = m \to n$.

Simplified proof:
(1) Each $m \in M$ corresponds to column $A(-, m)$,
(2) The "mth" column $A(-, m)$ is restored as $\delta_m \to A$ by row vector

$$\delta_m(n) = \begin{cases} e, & m = n, \\ 0, & m \neq n. \end{cases}$$

(3) $a \to (m \to n) = (am) \to n$.

Conclusion



Every module of a semigroup/monoid object of Set / Pos / Slat / Sup is a submodule of a matrix module over its enveloping unital quantale.