Monadic second order logic on infinite words is the model companion of linear temporal logic

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Introduction

- **$S1S$** (second-order logic of one successor) is a monadic second-order logic interpreted in the structure $(\omega, \leq, S, 0)$.

Büchi (1962) proved that $S1S$ is decidable. His proof uses a back-and-forth conversion between second-order formulas and automata on infinite words. If one converts a formula $\phi$ into an automaton, and then back into a formula, one obtains an equivalent formula $\phi'$ which is 'almost existential'. First-order theories in which every formula is equivalent to an existential one are called model complete.
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S1S is the model companion of LTL.
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- ... in a sense we make precise by introducing a suitable class of temporal algebras.
Model companion: definition

Definition

Let $T$ and $T^*$ be theories in a first-order language $\mathcal{L}$.
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1. $T^*$ is model complete, i.e., for any $\mathcal{L}$-formula $\phi$ there is an existential $\mathcal{L}$-formula $\phi'$ such that $T \vdash \phi \leftrightarrow \phi'$;
2. $T^*$ is a companion of $T$, i.e., for any universal $\mathcal{L}$-formula $\phi$, $T^* \vdash \phi$ if, and only if, $T \vdash \phi$.

Equivalently,

1. All embeddings between $T^*$-models are elementary;
2. Every $T$-model embeds in some $T^*$-model and vice versa.
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1. All embeddings between $T^*$-models are elementary;

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Fact

A theory $T^*$ is a model companion of $T$ iff the class of $T^*$-models coincides with the class of existentially closed $T$-models.
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- In particular, any theory $T$ has at most one model companion, and the model companion of $T$ exists iff the class of existentially closed $T$-models is elementary.
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- A model companion $T^*$ of $T$ is a model completion iff the class of $T$-models has amalgamation iff $T^*$ has quantifier elimination.
### Model companion: examples

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1. Integral domains
2. Algebraically closed fields
3. Linear orders
4. Dense linear orders without endpoints
5. Boolean algebras
6. Atomless Boolean algebras
7. Gödel algebras
8. Gödel algebras with density and splitting

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cf. L. Darnière & M. Junker, "Model completions of varieties of co-Heyting algebras".

(All of these examples are in fact model completions.)

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S1S is the model companion of LTL
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**LTL$_I$-algebras**

LTL$_I$-algebras are the universal class of BAO’s corresponding to linear temporal logic without until, enriched with an ‘initial atom’, $I$. 

Example

The complex algebra of $(\omega, \leq, 0, S)$ is an LTL$_I$-algebra $(\mathcal{P}(\omega), \cup, -, \emptyset, \diamond, X, I)$, where $\diamond a := \downarrow a$, $X a := S^{-1}(a)$, and $I := \{0\}$. 

**Definition**

An LTL$_I$-algebra is a tuple $(A, \cup, -, \emptyset, \diamond, X, I)$, where

1. $(A, \cup, -, \emptyset)$ is a Boolean algebra;
2. $\diamond : A \to A$ is a modal operator on $A$, i.e., preserves $\emptyset$ and $\cup$;
3. $X : A \to A$ is a Boolean endomorphism on $A$;
4. for any $a \in A$, the following conditions hold:
   1. $\diamond a = a \cup X \diamond a$,
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**LTL_1**-algebras

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The complex algebra of \((\omega, \leq, 0, S)\) is an **LTL_1**-algebra \((\mathcal{P}(\omega), \cup, -, \emptyset, \lozenge, X, I)\), where \( \lozenge a := \downarrow a \), \( Xa := S^{-1}(a) \), and \( I := \{0\} \).
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S1S is the model companion of LTL
Model companion of $\text{LTL}_I$-algebras

Let $\mathcal{L}$ be the first-order language with binary operation $\cup$, unary operations $\neg$, $\diamond$, and $\mathbf{X}$, and constant symbols $\emptyset$ and $\mathbf{I}$. 

Theorem

The theory of the $\mathcal{L}$-structure $P(\omega)$ is the model companion of the theory of $\text{LTL}_I$-algebras.

Proof outline.

Let $T$ be the theory of $\text{LTL}_I$-algebras and $T^*$ the theory of the $\mathcal{L}$-structure $P(\omega)$.

Two parts:

1. $T^*$ is a companion of $T$, via duality and filtrations;
2. $T^*$ is model complete, via automata.
Model companion of \( \text{LTL}_{I} \)-algebras

Let \( \mathcal{L} \) be the first-order language with binary operation \( \cup \), unary operations \( - \), \( \Diamond \), and \( \mathbf{X} \), and constant symbols \( \emptyset \) and \( I \).

**Theorem**

The theory of the \( \mathcal{L} \)-structure \( \mathcal{P}(\omega) \) is the model companion of the theory of \( \text{LTL}_{I} \)-algebras.
Model companion of $\text{LTL}_I$-algebras

Let $\mathcal{L}$ be the first-order language with binary operation $\cup$, unary operations $\neg$, $\diamond$, and $\mathbf{X}$, and constant symbols $\emptyset$ and $\mathbf{I}$.

**Theorem**

*The theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$ is the model companion of the theory of $\text{LTL}_I$-algebras.***

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Let $T$ be the theory of $\text{LTL}_I$-algebras and $T^*$ the theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$.

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**Theorem**

The theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$ is the model companion of the theory of $\text{LTL}_I$-algebras.

**Proof outline.**

Let $T$ be the theory of $\text{LTL}_I$-algebras and $T^\ast$ the theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$.

Two parts:

1. $T^\ast$ is a companion of $T$, via duality and filtrations;
Model companion of $\text{LTL}_1$-algebras

Let $\mathcal{L}$ be the first-order language with binary operation $\cup$, unary operations $\neg$, $\Diamond$, and $\mathbf{X}$, and constant symbols $\emptyset$ and $\mathbf{I}$.

Theorem

The theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$ is the model companion of the theory of $\text{LTL}_1$-algebras.

Proof outline.

Let $T$ be the theory of $\text{LTL}_1$-algebras and $T^*$ the theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$.

Two parts:

1. $T^*$ is a companion of $T$, via duality and filtrations;
2. $T^*$ is model complete, via automata.
Model companion of $\text{LTL}_{I}$-algebras

Let $\mathcal{L}$ be the first-order language with binary operation $\cup$, unary operations $\neg$, $\lozenge$, and $\Box$, and constant symbols $\emptyset$ and $I$.

**Theorem**

*The theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$ is the model companion of the theory of $\text{LTL}_{I}$-algebras.*

**Proof outline.**

Let $T$ be the theory of $\text{LTL}_{I}$-algebras and $T^*$ the theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$.

Two parts:

1. $T^*$ is a companion of $T$, via duality and filtrations;
2. $T^*$ is model complete, via automata.
Fact

For any \( \text{LTL}_{I} \)-algebra \((A, \cup, -, \emptyset, \Diamond, X, I)\),

- the reduct \((A, \cup, -, \emptyset, \Diamond)\) is an S4-algebra;
- the element \(I\) is an atom.

The Stone-Jónsson-Tarski dual of the \((\cup, -, \emptyset, \Diamond)\)-reduct of an \(\text{LTL}_{I}\)-algebra is a Boolean space \(X\) equipped with a preorder \(\leq\) that is compatible with the topology.

The dual of the Boolean homomorphism \(X\) is a continuous function, \(f\), on \(X\).

The dual of \(I\) is an isolated point, \(x_0\), of \(X\).

The axioms can also be translated, leading to a definition of \(\text{LTL}_{I}\)-space.
Duality for $\text{LTL}_I$-algebras

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For any $\text{LTL}_{I}$-algebra $(A, \cup, -, \emptyset, \diamond, X, I)$,

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2. the element $I$ is an atom.
Duality for LTL$_1$-algebras

Fact

For any LTL$_1$-algebra $(A, \cup, -, \emptyset, \lozenge, X, \mathbf{I})$,

1. the reduct $(A, \cup, -, \emptyset, \lozenge)$ is an S4-algebra;
2. the element $\mathbf{I}$ is an atom.

The Stone-Jónsson-Tarski dual of the $(\cup, -, \emptyset, \lozenge)$-reduct of an LTL$_1$-algebra is a Boolean space $X$ equipped with a preorder $\leq$ that is compatible with the topology.
Duality for $\text{LTL}_I$-algebras

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The dual of the Boolean homomorphism $X$ is a continuous function, $f$, on $X$. 
Duality for $\text{LTL}_I$-algebras

Fact

For any $\text{LTL}_I$-algebra $(A, \cup, -, \emptyset, \Diamond, X, 1)$,

1. the reduct $(A, \cup, -, \emptyset, \Diamond)$ is an S4-algebra;
2. the element $1$ is an atom.

- The Stone-Jónsson-Tarski dual of the $(\cup, -, \emptyset, \Diamond)$-reduct of an $\text{LTL}_I$-algebra is a Boolean space $X$ equipped with a preorder $\leq$ that is compatible with the topology.

- The dual of the Boolean homomorphism $X$ is a continuous function, $f$, on $X$.

- The dual of $1$ is an isolated point, $x_0$, of $X$. 

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Duality for LTL$_I$-algebras

Fact

For any LTL$_I$-algebra $(A, \cup, -, \emptyset, \Diamond, X, I)$,

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- The dual of the Boolean homomorphism $X$ is a continuous function, $f$, on $X$.
- The dual of $I$ is an isolated point, $x_0$, of $X$.
- The axioms can also be translated, leading to a definition of LTL$_I$-space.
Duality for $\text{LTL}_1$-algebras

**Definition**

An **$\text{LTL}_1$-space** is a tuple $(X, \leq, f, x_0)$, where

- $X$ is a Boolean topological space,
- $\leq$ is a preorder on $X$ that is compatible with the topology,
- $f : X \to X$ is a continuous function,
- $x_0 \in X$ is a point such that $\{x_0\}$ is open,

and for any $x, y \in X$ and clopen $K \subseteq X$:

1. $x \leq f(x)$ and $x < y \Rightarrow f(x) \leq y$,
2. if $f(K) \subseteq K$ then $\uparrow K \subseteq K$,
3. $x_0 \leq x$,
4. $f(x) \neq x_0$.
Duality for $\text{LTL}_I$-algebras

Definition

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Theorem

The category of $\text{LTL}_I$-algebras and homomorphisms is dually equivalent to the category of $\text{LTL}_I$-spaces and continuous $p$-morphisms.
Filtration for $\text{LTL}_1$-spaces

The duality can be used to prove:

Theorem (Completeness of $\text{LTL}_1$ with respect to $\omega$) For any $L$-term $t$, if $P(\omega) \models t = \emptyset$, then for any $\text{LTL}_1$-algebra $A$, $A \models t = \emptyset$. The crucial step in the proof of this theorem is a filtration lemma for $\text{LTL}_1$-spaces, adapting the usual completeness proof of $\text{LTL}$ to our setting.
Filtration for \( \text{LTL}_I \)-spaces

The duality can be used to prove:

**Theorem (Completeness of \( \text{LTL}_I \) with respect to \( \omega \))**

For any \( \mathcal{L} \)-term \( t \), if \( \mathcal{P}(\omega) \models t = \emptyset \), then for any \( \text{LTL}_I \)-algebra \( A \), \( A \models t = \emptyset \).
Filtration for $\text{LTL}_I$-spaces

The duality can be used to prove:

Theorem (Completeness of $\text{LTL}_I$ with respect to $\omega$)

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Companionship from completeness

Lemma

Any quantifier-free $\mathcal{L}$-formula is $T$-provably equivalent to a positive quantifier-free $\mathcal{L}$-formula.
Companionship from completeness

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Any quantifier-free $\mathcal{L}$-formula is $T$-provably equivalent to a positive quantifier-free $\mathcal{L}$-formula.

Proof.

The crucial thing to note is that, in any $\text{LTL}_I$-algebra,

$$a \neq \emptyset \text{ if, and only if, } I \subseteq \Diamond \neg a.$$
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Theorem

The theory of $\mathcal{P}(\omega)$ is a companion of the theory of $\text{LTL}_I$-algebras.
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By the Lemma and standard Boolean algebra facts, any universal $\mathcal{L}$-sentence is equivalent to one of the form $\forall \overline{v}(t(\overline{v}) = \emptyset)$.  

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The theory of $\mathcal{P}(\omega)$ is a companion of the theory of $\mathbf{LTL}_I$-algebras.

Proof.

- By the Lemma and standard Boolean algebra facts, any universal $\mathcal{L}$-sentence is equivalent to one of the form $\forall \overline{v}(t(\overline{v}) = \emptyset)$.
- By the Completeness Theorem, if this sentence is true in $\mathcal{P}(\omega)$, then it is true in all $\mathbf{LTL}_I$-algebras.
Model companion of $\text{LTL}_I$-algebras

Let $\mathcal{L}$ be the first-order language with binary operation $\cup$, unary operations $\neg$, $\Diamond$, and $X$, and constant symbols $\emptyset$ and $I$.

**Theorem**

The theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$ is the model companion of the theory of $\text{LTL}_I$-algebras.

**Proof outline.**

Let $T$ be the theory of $\text{LTL}_I$-algebras and $T^*$ the theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$.

Two parts:

1. $T^*$ is a companion of $T$, via duality and filtrations;
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Model companion of $\text{LTL}_I$-algebras

Let $\mathcal{L}$ be the first-order language with binary operation $\cup$, unary operations $-$, $\lozenge$, and $\textbf{X}$, and constant symbols $\emptyset$ and $I$.

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S1S-formulae are built up from atomic formulae $X \subseteq Y$ and $S(X, Y)$ with connectives $\lor$, $\neg$, and $\exists X$. 
**S1S**

- **S1S-formulae** are built up from atomic formulae $X \subseteq Y$ and $S(X, Y)$ with connectives $\lor$, $\neg$, and $\exists X$.
- **Valuations** $\bar{a} \in \mathcal{P}(\omega)^k$ correspond one-to-one to infinite words over the alphabet $\Sigma_k := \mathcal{P}(k)$, via the bijection:

$$\mathcal{P}(\omega)^k \cong (\Sigma_k)^\omega,$$

$$\bar{a} \mapsto w_{\bar{a}} : n \mapsto \{ i \in \{1, \ldots, k\} \mid n \in a_i \}.$$
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- Any S1S-formula $\varphi(X_1, \ldots, X_k)$ describes a set, or language, of infinite words over the alphabet $\Sigma_k$:
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- Any S1S-formula $\varphi(X_1, \ldots, X_k)$ describes a set, or language, of infinite words over the alphabet $\Sigma_k$:

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- For example:
  - the formula $\exists Y \neg (X_1 \subseteq Y)$ describes $\{ w \in 2^\omega | w(n) = 1 \text{ for some } n \in \omega \}$.
  - the (abbreviated) formula
    $\exists Y(X_1 \subseteq Y \land \text{zero}(Y) \land \forall Z(\text{singleton}(Z) \rightarrow (Z \subseteq Y \leftrightarrow \neg S(Z, Y)))$
    describes $\{ w \in 2^\omega | \text{if } w(n) = 1 \text{ then } n \text{ is even} \}$. 

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Büchi’s theorem

**Theorem (Büchi 1962)**

The languages of infinite words that can be described by S1S-formulae are exactly those which are accepted by non-deterministic finite automata with the Büchi acceptance condition.

Note: if \( A \) is an \( m \)-state automaton with set of final states \( F \subseteq \{1, \ldots, m\} \), and \( \rho \) is an infinite run of \( A \), then \( \rho \) is successful if, and only if, in the algebra \( \mathcal{P}(\omega) \), we have

\[
\omega = \bigcup_{i \in F} \diamond q_i,
\]

where \( q_i := \{ t \in \omega | \rho \text{ is in state } i \text{ at time } t \} \).

\((\star)\) is a quantifier-free formula in the first-order language \( L \).
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- Büchi acceptance condition: an infinite run of the automaton is successful iff there is a final state that occurs infinitely often.
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- Note: if $A$ is an $m$-state automaton with set of final states $F \subseteq \{1, \ldots, m\}$, and $\rho$ is an infinite run of $A$, then $\rho$ is successful if, and only if, in the algebra $P(\omega)$, we have

$$\omega = \bigcup_{i \in F} \Diamond q_i, \quad (\star)$$

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- ($\star$) is a quantifier-free formula in the first-order language $\mathcal{L}$!
Th(\mathcal{P}(\omega)) \text{ is model-complete}

- Via standard translation, we associate with any \( \mathcal{L} \)-formula \( \varphi(x_1, \ldots, x_n) \) an S1S-formula \( \hat{\varphi}(X_1, \ldots, X_n) \) such that, for any \( \bar{a} \in \mathcal{P}(\omega)^k \),

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(\mathcal{P}(\omega), \bar{a}) \models \varphi \iff \omega\bar{a} \in L_{\hat{\varphi}}.
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- By Büchi’s theorem, we pick an automaton \mathcal{A} in alphabet \Sigma_k recognizing \hat{L}_{\hat{\varphi}}.

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- Using \(\star\), we construct an \(\mathcal{L}\)-term \(t_{\mathcal{A}}\) which describes ‘successful runs of \(\mathcal{A}\)’.
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- The term $t_A$ yields an existential $\mathcal{L}$-formula which is equivalent to the formula $\varphi(x_1, \ldots, x_n)$ in the LTL$_1$-algebra $\mathcal{P}(\omega)$.
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- The term \( t_{\mathcal{A}} \) yields an existential \mathcal{L}\text{-formula which is equivalent to the formula } \varphi(x_1, \ldots, x_n) in the \text{LTL}_I\text{-algebra } \mathcal{P}(\omega).

- This completes the proof that \( T^* = \text{Th}(\mathcal{P}(\omega)) \) is model complete.
Model companion of $\text{LTL}_{I}$-algebras

Let $\mathcal{L}$ be the first-order language with binary operation $\cup$, unary operations $\neg$, $\Diamond$, and $X$, and constant symbols $\emptyset$ and $I$.

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The theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$ is the model companion of the theory of $\text{LTL}_{I}$-algebras.

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**Proposition**

*The theory of the algebra $P_{fc}(\omega)$ is the model companion of the theory of $\text{LTL}_I$-algebras satisfying the condition $\Box\Diamond x \leq \Diamond\Box x$.***
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- These results are just a first step, many questions remain:
  - Can a similar result be obtained for other structures, such as finite trees, infinite trees, ...?
  - What is the general mechanism underlying such results?
  - Can these results be used to give (alternative) axiomatizations of monadic second order logics?