# Monadic second order logic on infinite words is the model companion of linear temporal logic

Silvio Ghilardi and Sam van Gool

Università degli Studi di Milano

TACL, Ischia, 26 June 2015

# Introduction

- *S1S* (second-order logic of one successor) is a monadic second-order logic interpreted in the structure $(\omega, \leq, S, 0)$.

# Introduction

- *S1S* (second-order logic of one successor) is a monadic second-order logic interpreted in the structure $(\omega, \leq, S, 0)$.
- Büchi (1962) proved that S1S is decidable.

# Introduction

- *S1S* (second-order logic of one successor) is a monadic second-order logic interpreted in the structure $(\omega, \leq, S, 0)$.
- Büchi (1962) proved that S1S is decidable.
- His proof uses a back-and-forth conversion between second-order formulas and automata on infinite words.

# Introduction

- *S1S* (second-order logic of one successor) is a monadic second-order logic interpreted in the structure $(\omega, \leq, S, 0)$.
- Büchi (1962) proved that S1S is decidable.
- His proof uses a back-and-forth conversion between second-order formulas and automata on infinite words.
- If one converts a formula $\varphi$ into an automaton, and then back into a formula, one obtains an equivalent formula $\varphi'$ which is 'almost existential'.

# Introduction

- *S1S* (second-order logic of one successor) is a monadic second-order logic interpreted in the structure $(\omega, \leq, S, 0)$.
- Büchi (1962) proved that S1S is decidable.
- His proof uses a back-and-forth conversion between second-order formulas and automata on infinite words.
- If one converts a formula $\varphi$ into an automaton, and then back into a formula, one obtains an equivalent formula $\varphi'$ which is 'almost existential'.
- First-order theories in which every formula is equivalent to an existential one are called *model complete*.

# Introduction

How to make this connection to model completeness more explicit?

# Introduction

How to make this connection to model completeness more explicit?

- S1S can be read as a first-order logic, interpreted in $\mathcal{P}(\omega)$.

# Introduction

How to make this connection to model completeness more explicit?

- S1S can be read as a first-order logic, interpreted in $\mathcal{P}(\omega)$.
- We enrich the Boolean algebra $\mathcal{P}(\omega)$ with suitable modal operators, allowing us to convert an automaton with the Büchi acceptance condition into a genuine existential formula.

# Introduction

How to make this connection to model completeness more explicit?

- S1S can be read as a first-order logic, interpreted in $\mathcal{P}(\omega)$.
- We enrich the Boolean algebra $\mathcal{P}(\omega)$ with suitable modal operators, allowing us to convert an automaton with the Büchi acceptance condition into a genuine existential formula.
- In this way, S1S turns out to be the *model companion* of the temporal logic LTL ...

# Introduction

How to make this connection to model completeness more explicit?

- S1S can be read as a first-order logic, interpreted in $\mathcal{P}(\omega)$.
- We enrich the Boolean algebra $\mathcal{P}(\omega)$ with suitable modal operators, allowing us to convert an automaton with the Büchi acceptance condition into a genuine existential formula.
- In this way, S1S turns out to be the *model companion* of the temporal logic LTL ...
- ... in a sense we make precise by introducing a suitable class of temporal algebras.

# Model companion: definition

## Definition

Let $T$ and $T^*$ be theories in a first-order language $\mathcal{L}$.

# Model companion: definition

## Definition

Let $T$ and $T^*$ be theories in a first-order language $\mathcal{L}$.
The theory $T^*$ is called a model companion of $T$ provided that

# Model companion: definition

## Definition

Let $T$ and $T^*$ be theories in a first-order language $\mathcal{L}$.
The theory $T^*$ is called a model companion of $T$ provided that

1. $T^*$ is model complete, i.e., for any $\mathcal{L}$-formula $\varphi$ there is an existential $\mathcal{L}$-formula $\varphi'$ such that $T \vdash \varphi \leftrightarrow \varphi'$;

# Model companion: definition

## Definition

Let $T$ and $T^*$ be theories in a first-order language $\mathcal{L}$.
The theory $T^*$ is called a model companion of $T$ provided that

1. $T^*$ is model complete, i.e., for any $\mathcal{L}$-formula $\varphi$ there is an existential $\mathcal{L}$-formula $\varphi'$ such that $T \vdash \varphi \leftrightarrow \varphi'$;
2. $T^*$ is a companion of $T$, i.e., for any universal $\mathcal{L}$-formula $\varphi$, $T^* \vdash \varphi$ if, and only if, $T \vdash \varphi$.

# Model companion: definition

## Definition

Let $T$ and $T^*$ be theories in a first-order language $\mathcal{L}$.
The theory $T^*$ is called a model companion of $T$ provided that

1. $T^*$ is model complete, i.e., for any $\mathcal{L}$-formula $\varphi$ there is an existential $\mathcal{L}$-formula $\varphi'$ such that $T \vdash \varphi \leftrightarrow \varphi'$;
2. $T^*$ is a companion of $T$, i.e., for any universal $\mathcal{L}$-formula $\varphi$, $T^* \vdash \varphi$ if, and only if, $T \vdash \varphi$.

Equivalently,

# Model companion: definition

## Definition

Let $T$ and $T^*$ be theories in a first-order language $\mathcal{L}$.
The theory $T^*$ is called a model companion of $T$ provided that

1. $T^*$ is model complete, i.e., for any $\mathcal{L}$-formula $\varphi$ there is an existential $\mathcal{L}$-formula $\varphi'$ such that $T \vdash \varphi \leftrightarrow \varphi'$;
2. $T^*$ is a companion of $T$, i.e., for any universal $\mathcal{L}$-formula $\varphi$, $T^* \vdash \varphi$ if, and only if, $T \vdash \varphi$.

Equivalently,

1. All embeddings between $T^*$-models are elementary;

# Model companion: definition

## Definition

Let $T$ and $T^*$ be theories in a first-order language $\mathcal{L}$.
The theory $T^*$ is called a model companion of $T$ provided that

1. $T^*$ is model complete, i.e., for any $\mathcal{L}$-formula $\varphi$ there is an existential $\mathcal{L}$-formula $\varphi'$ such that $T \vdash \varphi \leftrightarrow \varphi'$;
2. $T^*$ is a companion of $T$, i.e., for any universal $\mathcal{L}$-formula $\varphi$, $T^* \vdash \varphi$ if, and only if, $T \vdash \varphi$.

Equivalently,

1. All embeddings between $T^*$-models are elementary;
2. Every $T$-model embeds in some $T^*$-model and vice versa.

# Model companion: basic facts

**Fact**

- *A theory $T^*$ is a model companion of $T$ iff the class of $T^*$-models coincides with the class of existentially closed $T$-models.*

# Model companion: basic facts

**Fact**

- *A theory $T^*$ is a model companion of $T$ iff the class of $T^*$-models coincides with the class of existentially closed $T$-models.*
- *In particular, any theory $T$ has at most one model companion, and the model companion of $T$ exists iff the class of existentially closed $T$-models is elementary.*

# Model companion: basic facts

## Fact

- *A theory $T^*$ is a model companion of $T$ iff the class of $T^*$-models coincides with the class of existentially closed $T$-models.*
- *In particular, any theory $T$ has at most one model companion, and the model companion of $T$ exists iff the class of existentially closed $T$-models is elementary.*
- *A model companion $T^*$ of $T$ is a model completion iff the class of $T$-models has amalgamation iff $T^*$ has quantifier elimination.*

# Model companion: examples

| Theory | Model companion |
| --- | --- |
| | |

# Model companion: examples

| Theory | Model companion |
|---|---|
| Integral domains | Algebraically closed fields |
| | |

# Model companion: examples

| Theory | Model companion |
|---|---|
| Integral domains | Algebraically closed fields |
| Linear orders | Dense linear orders without endpoints |
| | |

# Model companion: examples

| Theory | Model companion |
|---|---|
| Integral domains | Algebraically closed fields |
| Linear orders | Dense linear orders without endpoints |
| Boolean algebras | Atomless Boolean algebras |
| | |

# Model companion: examples

| Theory | Model companion |
|---|---|
| Integral domains | Algebraically closed fields |
| Linear orders | Dense linear orders without endpoints |
| Boolean algebras | Atomless Boolean algebras |
| Gödel algebras | Gödel algebras with density and splitting[1] |

# Model companion: examples

| Theory | Model companion |
| --- | --- |
| Integral domains | Algebraically closed fields |
| Linear orders | Dense linear orders without endpoints |
| Boolean algebras | Atomless Boolean algebras |
| Gödel algebras | Gödel algebras with density and splitting[1] |

[1] cf. L. Darnière & M. Junker, "Model completions of varieties of co-Heyting algebras".

# Model companion: examples

| Theory | Model companion |
|---|---|
| Integral domains | Algebraically closed fields |
| Linear orders | Dense linear orders without endpoints |
| Boolean algebras | Atomless Boolean algebras |
| Gödel algebras | Gödel algebras with density and splitting[1] |

[1] cf. L. Darnière & M. Junker, “Model completions of varieties of co-Heyting algebras”.

(All of these examples are in fact model completions.)

# $\mathrm{LTL_I}$-algebras

$\mathrm{LTL_I}$-algebras are the universal class of BAO's corresponding to linear temporal logic without until, enriched with an 'initial atom', **I**.

# $\mathrm{LTL_I}$-algebras

$\mathrm{LTL_I}$-algebras are the universal class of BAO's corresponding to linear temporal logic without until, enriched with an 'initial atom', **I**.

## Example

The complex algebra of $(\omega, \leq, 0, S)$ is an $\mathrm{LTL_I}$-algebra $(\mathcal{P}(\omega), \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$, where $\Diamond a := \downarrow a$, $\mathbf{X}a := S^{-1}(a)$, and $\mathbf{I} := \{0\}$.

# $\mathrm{LTL_I}$-algebras

$\mathrm{LTL_I}$-algebras are the universal class of BAO's corresponding to linear temporal logic without until, enriched with an 'initial atom', **I**.

## Example

The complex algebra of $(\omega, \leq, 0, S)$ is an $\mathrm{LTL_I}$-algebra $(\mathcal{P}(\omega), \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$, where $\Diamond a := {\downarrow} a$, $\mathbf{X}a := S^{-1}(a)$, and $\mathbf{I} := \{0\}$.

## Definition

# $\mathrm{LTL_I}$-algebras

$\mathrm{LTL_I}$-algebras are the universal class of BAO's corresponding to linear temporal logic without until, enriched with an 'initial atom', **I**.

## Example

The complex algebra of $(\omega, \leq, 0, S)$ is an $\mathrm{LTL_I}$-algebra $(\mathcal{P}(\omega), \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$, where $\Diamond a := \downarrow a$, $\mathbf{X}a := S^{-1}(a)$, and $\mathbf{I} := \{0\}$.

## Definition

An $\mathrm{LTL_I}$*-algebra* is a tuple $(A, \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$, where

# $\mathrm{LTL_I}$-algebras

$\mathrm{LTL_I}$-algebras are the universal class of BAO's corresponding to linear temporal logic without until, enriched with an 'initial atom', **I**.

## Example

The complex algebra of $(\omega, \leq, 0, S)$ is an $\mathrm{LTL_I}$-algebra $(\mathcal{P}(\omega), \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$, where $\Diamond a := \downarrow a$, $\mathbf{X}a := S^{-1}(a)$, and $\mathbf{I} := \{0\}$.

## Definition

An $\mathrm{LTL_I}$*-algebra* is a tuple $(A, \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$, where

1. $(A, \cup, -, \emptyset)$ is a Boolean algebra;

# $\mathrm{LTL_I}$-algebras

$\mathrm{LTL_I}$-algebras are the universal class of BAO's corresponding to linear temporal logic without until, enriched with an 'initial atom', **I**.

## Example

The complex algebra of $(\omega, \leq, 0, S)$ is an $\mathrm{LTL_I}$-algebra $(\mathcal{P}(\omega), \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$, where $\Diamond a := \downarrow a$, $\mathbf{X}a := S^{-1}(a)$, and $\mathbf{I} := \{0\}$.

## Definition

An $\mathrm{LTL_I}$-*algebra* is a tuple $(A, \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$, where

1. $(A, \cup, -, \emptyset)$ is a Boolean algebra;
2. $\Diamond : A \to A$ is a modal operator on $A$, i.e., preserves $\emptyset$ and $\cup$;

# $\mathrm{LTL_I}$-algebras

$\mathrm{LTL_I}$-algebras are the universal class of BAO's corresponding to linear temporal logic without until, enriched with an 'initial atom', **I**.

## Example

The complex algebra of $(\omega, \leq, 0, S)$ is an $\mathrm{LTL_I}$-algebra $(\mathcal{P}(\omega), \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$, where $\Diamond a := \downarrow a$, $\mathbf{X}a := S^{-1}(a)$, and $\mathbf{I} := \{0\}$.

## Definition

An $\mathrm{LTL_I}$*-algebra* is a tuple $(A, \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$, where

1. $(A, \cup, -, \emptyset)$ is a Boolean algebra;
2. $\Diamond : A \to A$ is a modal operator on $A$, i.e., preserves $\emptyset$ and $\cup$;
3. $\mathbf{X} : A \to A$ is a Boolean endomorphism on $A$;

# $\mathrm{LTL_I}$-algebras

$\mathrm{LTL_I}$-algebras are the universal class of BAO's corresponding to linear temporal logic without until, enriched with an 'initial atom', **I**.

## Example

The complex algebra of $(\omega, \leq, 0, S)$ is an $\mathrm{LTL_I}$-algebra $(\mathcal{P}(\omega), \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$, where $\Diamond a := {\downarrow}a$, $\mathbf{X}a := S^{-1}(a)$, and $\mathbf{I} := \{0\}$.

## Definition

An $\mathrm{LTL_I}$-*algebra* is a tuple $(A, \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$, where

1. $(A, \cup, -, \emptyset)$ is a Boolean algebra;
2. $\Diamond : A \to A$ is a modal operator on $A$, i.e., preserves $\emptyset$ and $\cup$;
3. $\mathbf{X} : A \to A$ is a Boolean endomorphism on $A$;
4. for any $a \in A$, the following conditions hold:

# $\mathrm{LTL_I}$-algebras

$\mathrm{LTL_I}$-algebras are the universal class of BAO's corresponding to linear temporal logic without until, enriched with an 'initial atom', **I**.

## Example

The complex algebra of $(\omega, \leq, 0, S)$ is an $\mathrm{LTL_I}$-algebra $(\mathcal{P}(\omega), \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$, where $\Diamond a := \downarrow a$, $\mathbf{X}a := S^{-1}(a)$, and $\mathbf{I} := \{0\}$.

## Definition

An $\mathrm{LTL_I}$-*algebra* is a tuple $(A, \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$, where

1. $(A, \cup, -, \emptyset)$ is a Boolean algebra;
2. $\Diamond : A \to A$ is a modal operator on $A$, i.e., preserves $\emptyset$ and $\cup$;
3. $\mathbf{X} : A \to A$ is a Boolean endomorphism on $A$;
4. for any $a \in A$, the following conditions hold:
    1. $\Diamond a = a \cup \mathbf{X}\Diamond a$,

# $\mathrm{LTL_I}$-algebras

$\mathrm{LTL_I}$-algebras are the universal class of BAO's corresponding to linear temporal logic without until, enriched with an 'initial atom', **I**.

## Example

The complex algebra of $(\omega, \leq, 0, S)$ is an $\mathrm{LTL_I}$-algebra $(\mathcal{P}(\omega), \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$, where $\Diamond a := \downarrow a$, $\mathbf{X}a := S^{-1}(a)$, and $\mathbf{I} := \{0\}$.

## Definition

An $\mathrm{LTL_I}$*-algebra* is a tuple $(A, \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$, where

1. $(A, \cup, -, \emptyset)$ is a Boolean algebra;
2. $\Diamond : A \to A$ is a modal operator on $A$, i.e., preserves $\emptyset$ and $\cup$;
3. $\mathbf{X} : A \to A$ is a Boolean endomorphism on $A$;
4. for any $a \in A$, the following conditions hold:
    1. $\Diamond a = a \cup \mathbf{X}\Diamond a$,
    2. if $\mathbf{X}a \subseteq a$ then $\Diamond a \subseteq a$,

# $\mathrm{LTL_I}$-algebras

$\mathrm{LTL_I}$-algebras are the universal class of BAO's corresponding to linear temporal logic without until, enriched with an 'initial atom', **I**.

## Example

The complex algebra of $(\omega, \leq, 0, S)$ is an $\mathrm{LTL_I}$-algebra $(\mathcal{P}(\omega), \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$, where $\Diamond a := \downarrow a$, $\mathbf{X}a := S^{-1}(a)$, and $\mathbf{I} := \{0\}$.

## Definition

An $\mathrm{LTL_I}$*-algebra* is a tuple $(A, \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$, where

1. $(A, \cup, -, \emptyset)$ is a Boolean algebra;
2. $\Diamond : A \to A$ is a modal operator on $A$, i.e., preserves $\emptyset$ and $\cup$;
3. $\mathbf{X} : A \to A$ is a Boolean endomorphism on $A$;
4. for any $a \in A$, the following conditions hold:
    1. $\Diamond a = a \cup \mathbf{X}\Diamond a$,
    2. if $\mathbf{X}a \subseteq a$ then $\Diamond a \subseteq a$,
    3. if $a \neq \emptyset$ then $\mathbf{I} \subseteq \Diamond a$.

# $\mathrm{LTL_I}$-algebras

$\mathrm{LTL_I}$-algebras are the universal class of BAO's corresponding to linear temporal logic without until, enriched with an 'initial atom', **I**.

## Example

The complex algebra of $(\omega, \leq, 0, S)$ is an $\mathrm{LTL_I}$-algebra $(\mathcal{P}(\omega), \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$, where $\Diamond a := \downarrow a$, $\mathbf{X}a := S^{-1}(a)$, and $\mathbf{I} := \{0\}$.

## Definition

An $\mathrm{LTL_I}$-*algebra* is a tuple $(A, \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$, where

1. $(A, \cup, -, \emptyset)$ is a Boolean algebra;
2. $\Diamond : A \to A$ is a modal operator on $A$, i.e., preserves $\emptyset$ and $\cup$;
3. $\mathbf{X} : A \to A$ is a Boolean endomorphism on $A$;
4. for any $a \in A$, the following conditions hold:
   1. $\Diamond a = a \cup \mathbf{X}\Diamond a$,
   2. if $\mathbf{X}a \subseteq a$ then $\Diamond a \subseteq a$,
   3. if $a \neq \emptyset$ then $\mathbf{I} \subseteq \Diamond a$.
   4. $\mathbf{XI} = \emptyset$.

# Model companion of $\mathrm{LTL}_{\mathrm{I}}$-algebras

Let $\mathcal{L}$ be the first-order language with binary operation $\cup$, unary operations $-$, $\Diamond$, and $\mathbf{X}$, and constant symbols $\emptyset$ and $\mathbf{I}$.

# Model companion of $\mathrm{LTL_I}$-algebras

Let $\mathcal{L}$ be the first-order language with binary operation $\cup$, unary operations $-$, $\Diamond$, and $\mathbf{X}$, and constant symbols $\emptyset$ and $\mathbf{I}$.

**Theorem**

*The theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$ is the model companion of the theory of $\mathrm{LTL_I}$-algebras.*

# Model companion of $\mathrm{LTL_I}$-algebras

Let $\mathcal{L}$ be the first-order language with binary operation $\cup$, unary operations $-$, $\Diamond$, and $\mathbf{X}$, and constant symbols $\emptyset$ and $\mathbf{I}$.

**Theorem**

*The theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$ is the model companion of the theory of $\mathrm{LTL_I}$-algebras.*

**Proof outline.**

Let $T$ be the theory of $\mathrm{LTL_I}$-algebras and $T^*$ the theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$.
Two parts:

# Model companion of $\mathrm{LTL_I}$-algebras

Let $\mathcal{L}$ be the first-order language with binary operation $\cup$, unary operations $-$, $\Diamond$, and $\mathbf{X}$, and constant symbols $\emptyset$ and $\mathbf{I}$.

## Theorem

*The theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$ is the model companion of the theory of $\mathrm{LTL_I}$-algebras.*

## Proof outline.

Let $T$ be the theory of $\mathrm{LTL_I}$-algebras and $T^*$ the theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$.
Two parts:

1. $T^*$ is a companion of $T$, via duality and filtrations;

# Model companion of $\mathrm{LTL_I}$-algebras

Let $\mathcal{L}$ be the first-order language with binary operation $\cup$, unary operations $-$, $\Diamond$, and $\mathbf{X}$, and constant symbols $\emptyset$ and $\mathbf{I}$.

## Theorem

*The theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$ is the model companion of the theory of $\mathrm{LTL_I}$-algebras.*

## Proof outline.

Let $T$ be the theory of $\mathrm{LTL_I}$-algebras and $T^*$ the theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$.

Two parts:

1. $T^*$ is a companion of $T$, via duality and filtrations;
2. $T^*$ is model complete, via automata.

# Model companion of $\mathrm{LTL_I}$-algebras

Let $\mathcal{L}$ be the first-order language with binary operation $\cup$, unary operations $-$, $\Diamond$, and $\mathbf{X}$, and constant symbols $\emptyset$ and $\mathbf{I}$.

## Theorem

*The theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$ is the model companion of the theory of $\mathrm{LTL_I}$-algebras.*

## Proof outline.

Let $T$ be the theory of $\mathrm{LTL_I}$-algebras and $T^*$ the theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$.
Two parts:

1. $T^*$ is a companion of $T$, via duality and filtrations;
2. $T^*$ is model complete, via automata.

# Duality for $\mathrm{LTL_I}$-algebras

**Fact**

*For any* $\mathrm{LTL_I}$*-algebra* $(A, \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$,

# Duality for $\mathrm{LTL_I}$-algebras

## Fact

*For any $\mathrm{LTL_I}$-algebra $(A, \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$,*

1. *the reduct $(A, \cup, -, \emptyset, \Diamond)$ is an S4-algebra;*

# Duality for $\mathrm{LTL_I}$-algebras

**Fact**

*For any* $\mathrm{LTL_I}$*-algebra* $(A, \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$*,*

1. *the reduct* $(A, \cup, -, \emptyset, \Diamond)$ *is an S4-algebra;*
2. *the element* $\mathbf{I}$ *is an atom.*

# Duality for $\mathrm{LTL_I}$-algebras

## Fact

*For any* $\mathrm{LTL_I}$*-algebra* $(A, \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$,

1. *the reduct* $(A, \cup, -, \emptyset, \Diamond)$ *is an S4-algebra;*
2. *the element* $\mathbf{I}$ *is an atom.*

- The Stone-Jónsson-Tarski dual of the $(\cup, -, \emptyset, \Diamond)$-reduct of an $\mathrm{LTL_I}$-algebra is a Boolean space $X$ equipped with a preorder $\leq$ that is compatible with the topology.

# Duality for $\mathrm{LTL_I}$-algebras

**Fact**

*For any $\mathrm{LTL_I}$-algebra $(A, \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$,*

1. *the reduct $(A, \cup, -, \emptyset, \Diamond)$ is an S4-algebra;*
2. *the element $\mathbf{I}$ is an atom.*

- The Stone-Jónsson-Tarski dual of the $(\cup, -, \emptyset, \Diamond)$-reduct of an $\mathrm{LTL_I}$-algebra is a Boolean space $X$ equipped with a preorder $\leq$ that is compatible with the topology.
- The dual of the Boolean homomorphism **X** is a continuous function, $f$, on $X$.

# Duality for $\mathrm{LTL_I}$-algebras

**Fact**

*For any* $\mathrm{LTL_I}$*-algebra* $(A, \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$,

1. *the reduct* $(A, \cup, -, \emptyset, \Diamond)$ *is an S4-algebra;*
2. *the element* $\mathbf{I}$ *is an atom.*

- The Stone-Jónsson-Tarski dual of the $(\cup, -, \emptyset, \Diamond)$-reduct of an $\mathrm{LTL_I}$-algebra is a Boolean space $X$ equipped with a preorder $\leq$ that is compatible with the topology.
- The dual of the Boolean homomorphism $\mathbf{X}$ is a continuous function, $f$, on $X$.
- The dual of $\mathbf{I}$ is an isolated point, $x_0$, of $X$.

# Duality for $\mathrm{LTL_I}$-algebras

**Fact**

*For any $\mathrm{LTL_I}$-algebra $(A, \cup, -, \emptyset, \Diamond, \mathbf{X}, \mathbf{I})$,*

1. *the reduct $(A, \cup, -, \emptyset, \Diamond)$ is an S4-algebra;*
2. *the element $\mathbf{I}$ is an atom.*

- The Stone-Jónsson-Tarski dual of the $(\cup, -, \emptyset, \Diamond)$-reduct of an $\mathrm{LTL_I}$-algebra is a Boolean space $X$ equipped with a preorder $\leq$ that is compatible with the topology.
- The dual of the Boolean homomorphism $\mathbf{X}$ is a continuous function, $f$, on $X$.
- The dual of $\mathbf{I}$ is an isolated point, $x_0$, of $X$.
- The axioms can also be translated, leading to a definition of $\mathrm{LTL_I}$-space.

# Duality for $\mathrm{LTL_I}$-algebras

## Definition

An $\mathrm{LTL_I}$-*space* is a tuple $(X, \leq, f, x_0)$, where

- $X$ is a Boolean topological space,
- $\leq$ is a preorder on $X$ that is compatible with the topology,
- $f : X \to X$ is a continuous function,
- $x_0 \in X$ is a point such that $\{x_0\}$ is open,

and for any $x, y \in X$ and clopen $K \subseteq X$:

1. $x \leq f(x)$ and $x < y \Rightarrow f(x) \leq y$,
2. if $f(K) \subseteq K$ then $\uparrow K \subseteq K$,
3. $x_0 \leq x$,
4. $f(x) \neq x_0$.

# Duality for $\mathrm{LTL_I}$-algebras

## Definition

An $\mathrm{LTL_I}$-*space* is a tuple $(X, \leq, f, x_0)$, where

- $X$ is a Boolean topological space,
- $\leq$ is a preorder on $X$ that is compatible with the topology,
- $f : X \to X$ is a continuous function,
- $x_0 \in X$ is a point such that $\{x_0\}$ is open,

and for any $x, y \in X$ and clopen $K \subseteq X$:

1. $x \leq f(x)$ and $x < y \Rightarrow f(x) \leq y$,
2. if $f(K) \subseteq K$ then $\uparrow K \subseteq K$,
3. $x_0 \leq x$,
4. $f(x) \neq x_0$.

## Theorem

*The category of $\mathrm{LTL_I}$-algebras and homomorphisms is dually equivalent to the category of $\mathrm{LTL_I}$-spaces and continuous p-morphisms.*

# Filtration for $\mathrm{LTL}_{\mathrm{I}}$-spaces

The duality can be used to prove:

# Filtration for $\mathrm{LTL_I}$-spaces

The duality can be used to prove:

**Theorem (Completeness of $\mathrm{LTL_I}$ with respect to $\omega$)**

*For any $\mathcal{L}$-term $t$, if $\mathcal{P}(\omega) \models t = \emptyset$, then for any $\mathrm{LTL_I}$-algebra $\mathbb{A}$, $\mathbb{A} \models t = \emptyset$.*

# Filtration for $\mathrm{LTL_I}$-spaces

The duality can be used to prove:

**Theorem (Completeness of $\mathrm{LTL_I}$ with respect to $\omega$)**

*For any $\mathcal{L}$-term $t$, if $\mathcal{P}(\omega) \models t = \emptyset$, then for any $\mathrm{LTL_I}$-algebra $\mathbb{A}$, $\mathbb{A} \models t = \emptyset$.*

The crucial step in the proof of this theorem is a filtration lemma for $\mathrm{LTL_I}$-spaces, adapting the usual completeness proof of $\mathrm{LTL}$ to our setting.

# Companionship from completeness

**Lemma**

*Any quantifier-free $\mathcal{L}$-formula is $T$-provably equivalent to a positive quantifier-free $\mathcal{L}$-formula.*

# Companionship from completeness

**Lemma**

*Any quantifier-free $\mathcal{L}$-formula is $T$-provably equivalent to a positive quantifier-free $\mathcal{L}$-formula.*

**Proof.**

The crucial thing to note is that, in any $\mathrm{LTL_I}$-algebra,

$$a \neq \emptyset \text{ if, and only if, } \mathbf{I} \subseteq \Diamond - a. \qquad \square$$

# Companionship from completeness

## Lemma

*Any quantifier-free $\mathcal{L}$-formula is $T$-provably equivalent to a positive quantifier-free $\mathcal{L}$-formula.*

## Proof.

The crucial thing to note is that, in any $\mathrm{LTL_I}$-algebra,

$$a \neq \emptyset \text{ if, and only if, } \mathbf{I} \subseteq \Diamond{-a}. \qquad \square$$

## Theorem

*The theory of $\mathcal{P}(\omega)$ is a companion of the theory of $\mathrm{LTL_I}$-algebras.*

# Companionship from completeness

## Lemma

*Any quantifier-free $\mathcal{L}$-formula is $T$-provably equivalent to a positive quantifier-free $\mathcal{L}$-formula.*

## Proof.

The crucial thing to note is that, in any $\mathrm{LTL_I}$-algebra,

$$a \neq \emptyset \text{ if, and only if, } \mathbf{I} \subseteq \Diamond{-a}. \qquad \square$$

## Theorem

*The theory of $\mathcal{P}(\omega)$ is a companion of the theory of $\mathrm{LTL_I}$-algebras.*

## Proof.

# Companionship from completeness

**Lemma**

*Any quantifier-free $\mathcal{L}$-formula is $T$-provably equivalent to a positive quantifier-free $\mathcal{L}$-formula.*

**Proof.**

The crucial thing to note is that, in any $\mathrm{LTL_I}$-algebra,

$$a \neq \emptyset \text{ if, and only if, } \mathbf{I} \subseteq \Diamond{-a}. \qquad \square$$

**Theorem**

*The theory of $\mathcal{P}(\omega)$ is a companion of the theory of $\mathrm{LTL_I}$-algebras.*

**Proof.**

- By the Lemma and standard Boolean algebra facts, any universal $\mathcal{L}$-sentence is equivalent to one of the form $\forall \overline{v}(t(\overline{v}) = \emptyset)$.

# Companionship from completeness

## Lemma

*Any quantifier-free $\mathcal{L}$-formula is $T$-provably equivalent to a positive quantifier-free $\mathcal{L}$-formula.*

## Proof.

The crucial thing to note is that, in any $\mathrm{LTL_I}$-algebra,

$$a \neq \emptyset \text{ if, and only if, } \mathbf{I} \subseteq \Diamond{-a}.$$

□

## Theorem

*The theory of $\mathcal{P}(\omega)$ is a companion of the theory of $\mathrm{LTL_I}$-algebras.*

## Proof.

- By the Lemma and standard Boolean algebra facts, any universal $\mathcal{L}$-sentence is equivalent to one of the form $\forall \overline{v}(t(\overline{v}) = \emptyset)$.
- By the Completeness Theorem, if this sentence is true in $\mathcal{P}(\omega)$, then it is true in all $\mathrm{LTL_I}$-algebras.

□

# Model companion of $\mathrm{LTL_I}$-algebras

Let $\mathcal{L}$ be the first-order language with binary operation $\cup$, unary operations $-$, $\Diamond$, and $\mathbf{X}$, and constant symbols $\emptyset$ and $\mathbf{I}$.

## Theorem

*The theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$ is the model companion of the theory of $\mathrm{LTL_I}$-algebras.*

## Proof outline.

Let $T$ be the theory of $\mathrm{LTL_I}$-algebras and $T^*$ the theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$.
Two parts:

1. $T^*$ is a companion of $T$, via duality and filtrations;
2. $T^*$ is model complete, via automata.

# Model companion of $\mathrm{LTL_I}$-algebras

Let $\mathcal{L}$ be the first-order language with binary operation $\cup$, unary operations $-$, $\Diamond$, and **X**, and constant symbols $\emptyset$ and **I**.

## Theorem

*The theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$ is the model companion of the theory of $\mathrm{LTL_I}$-algebras.*

## Proof outline.

Let $T$ be the theory of $\mathrm{LTL_I}$-algebras and $T^*$ the theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$.
Two parts:

1. $T^*$ is a companion of $T$, via duality and filtrations;
2. $T^*$ is model complete, via automata.

# S1S

- S1S-formulae are built up from atomic formulae $X \subseteq Y$ and $S(X, Y)$ with connectives $\vee$, $\neg$, and $\exists X$.

# S1S

- S1S-formulae are built up from atomic formulae $X \subseteq Y$ and $S(X, Y)$ with connectives $\vee$, $\neg$, and $\exists X$.
- Valuations $\overline{a} \in \mathcal{P}(\omega)^k$ correspond one-to-one to infinite words over the alphabet $\Sigma_k := \mathcal{P}(k)$, via the bijection:

$$\mathcal{P}(\omega)^k \cong (\Sigma_k)^\omega,$$
$$\overline{a} \mapsto w_{\overline{a}} : n \mapsto \{i \in \{1, \dots, k\} \mid n \in a_i\}.$$

# S1S

- S1S-formulae are built up from atomic formulae $X \subseteq Y$ and $S(X, Y)$ with connectives $\vee$, $\neg$, and $\exists X$.
- Valuations $\overline{a} \in \mathcal{P}(\omega)^k$ correspond one-to-one to infinite words over the alphabet $\Sigma_k := \mathcal{P}(k)$, via the bijection:

$$\mathcal{P}(\omega)^k \cong (\Sigma_k)^\omega,$$
$$\overline{a} \mapsto w_{\overline{a}} : n \mapsto \{i \in \{1, \ldots, k\} \mid n \in a_i\}.$$

- Any S1S-formula $\varphi(X_1, \ldots, X_k)$ describes a set, or language, of infinite words over the alphabet $\Sigma_k$:

$$L_\varphi := \{w_{\overline{a}} \mid \varphi[X_i \mapsto a_i] \text{ is true}\}.$$

# S1S

- S1S-formulae are built up from atomic formulae $X \subseteq Y$ and $S(X, Y)$ with connectives $\vee$, $\neg$, and $\exists X$.
- Valuations $\overline{a} \in \mathcal{P}(\omega)^k$ correspond one-to-one to infinite words over the alphabet $\Sigma_k := \mathcal{P}(k)$, via the bijection:

$$\mathcal{P}(\omega)^k \cong (\Sigma_k)^\omega,$$
$$\overline{a} \mapsto w_{\overline{a}} : n \mapsto \{i \in \{1, \ldots, k\} \mid n \in a_i\}.$$

- Any S1S-formula $\varphi(X_1, \ldots, X_k)$ describes a set, or language, of infinite words over the alphabet $\Sigma_k$:

$$L_\varphi := \{w_{\overline{a}} \mid \varphi[X_i \mapsto a_i] \text{ is true}\}.$$

- For example:

# S1S

- S1S-formulae are built up from atomic formulae $X \subseteq Y$ and $S(X, Y)$ with connectives $\vee$, $\neg$, and $\exists X$.
- Valuations $\overline{a} \in \mathcal{P}(\omega)^k$ correspond one-to-one to infinite words over the alphabet $\Sigma_k := \mathcal{P}(k)$, via the bijection:

$$\mathcal{P}(\omega)^k \cong (\Sigma_k)^\omega,$$
$$\overline{a} \mapsto w_{\overline{a}} : n \mapsto \{i \in \{1, \ldots, k\} \mid n \in a_i\}.$$

- Any S1S-formula $\varphi(X_1, \ldots, X_k)$ describes a set, or language, of infinite words over the alphabet $\Sigma_k$:

$$L_\varphi := \{w_{\overline{a}} \mid \varphi[X_i \mapsto a_i] \text{ is true}\}.$$

- For example:
  - the formula $\exists Y \neg(X_1 \subseteq Y)$ describes $\{w \in 2^\omega \mid w(n) = 1 \text{ for some } n \in \omega\}$.

# S1S

- S1S-formulae are built up from atomic formulae $X \subseteq Y$ and $S(X, Y)$ with connectives $\vee$, $\neg$, and $\exists X$.
- Valuations $\overline{a} \in \mathcal{P}(\omega)^k$ correspond one-to-one to infinite words over the alphabet $\Sigma_k := \mathcal{P}(k)$, via the bijection:

$$\mathcal{P}(\omega)^k \cong (\Sigma_k)^\omega,$$
$$\overline{a} \mapsto w_{\overline{a}} : n \mapsto \{i \in \{1, \ldots, k\} \mid n \in a_i\}.$$

- Any S1S-formula $\varphi(X_1, \ldots, X_k)$ describes a set, or language, of infinite words over the alphabet $\Sigma_k$:

$$L_\varphi := \{w_{\overline{a}} \mid \varphi[X_i \mapsto a_i] \text{ is true}\}.$$

- For example:
  - the formula $\exists Y \neg(X_1 \subseteq Y)$ describes $\{w \in 2^\omega \mid w(n) = 1 \text{ for some } n \in \omega\}$.
  - the (abbreviated) formula $\exists Y(X_1 \subseteq Y \wedge \mathbf{zero}(Y) \wedge \forall Z(\mathbf{singleton}(Z) \to (Z \subseteq Y \leftrightarrow \neg S(Z, Y)))$ describes $\{w \in 2^\omega \mid \text{if } w(n) = 1 \text{ then } n \text{ is even}\}$.

# Büchi's theorem

## Theorem (Büchi 1962)

*The languages of infinite words that can be described by S1S-formulae are exactly those which are accepted by non-deterministic finite automata with the Büchi acceptance condition.*

# Büchi's theorem

**Theorem (Büchi 1962)**

*The languages of infinite words that can be described by S1S-formulae are exactly those which are accepted by non-deterministic finite automata with the Büchi acceptance condition.*

- Büchi acceptance condition: an infinite run of the automaton is successful iff there is a final state that occurs infinitely often.

# Büchi's theorem

## Theorem (Büchi 1962)

*The languages of infinite words that can be described by S1S-formulae are exactly those which are accepted by non-deterministic finite automata with the Büchi acceptance condition.*

- Büchi acceptance condition: an infinite run of the automaton is successful iff there is a final state that occurs infinitely often.
- Note: if $\mathcal{A}$ is an $m$-state automaton with set of final states $F \subseteq \{1, \ldots, m\}$, and $\rho$ is an infinite run of $\mathcal{A}$, then $\rho$ is successful if, and only if, in the algebra $\mathcal{P}(\omega)$, we have

$$\omega = \bigcup_{i \in F} \Diamond q_i, \quad (\star)$$

where

$$q_i := \{t \in \omega \mid \rho \text{ is in state } i \text{ at time } t\}.$$

# Büchi's theorem

**Theorem (Büchi 1962)**

*The languages of infinite words that can be described by S1S-formulae are exactly those which are accepted by non-deterministic finite automata with the Büchi acceptance condition.*

- Büchi acceptance condition: an infinite run of the automaton is successful iff there is a final state that occurs infinitely often.
- Note: if $\mathcal{A}$ is an $m$-state automaton with set of final states $F \subseteq \{1, \ldots, m\}$, and $\rho$ is an infinite run of $\mathcal{A}$, then $\rho$ is successful if, and only if, in the algebra $\mathcal{P}(\omega)$, we have

$$\omega = \bigcup_{i \in F} \Diamond q_i, \quad (\star)$$

  where

$$q_i := \{t \in \omega \mid \rho \text{ is in state } i \text{ at time } t\}.$$

- $(\star)$ is a quantifier-free formula in the first-order language $\mathcal{L}$!

# $\mathrm{Th}(\mathcal{P}(\omega))$ is model-complete

- Via standard translation, we associate with any $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ an S1S-formula $\widehat{\varphi}(X_1, \ldots, X_n)$ such that, for any $\overline{a} \in \mathcal{P}(\omega)^k$,
$$(\mathcal{P}(\omega), \overline{a}) \models \varphi \iff w_{\overline{a}} \in L_{\widehat{\varphi}}.$$

# $\mathrm{Th}(\mathcal{P}(\omega))$ is model-complete

- Via standard translation, we associate with any $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ an S1S-formula $\widehat{\varphi}(X_1, \ldots, X_n)$ such that, for any $\overline{a} \in \mathcal{P}(\omega)^k$,
$$(\mathcal{P}(\omega), \overline{a}) \models \varphi \iff w_{\overline{a}} \in L_{\widehat{\varphi}}.$$
- By Büchi's theorem, we pick an automaton $\mathcal{A}$ in alphabet $\Sigma_k$ recognizing $L_{\widehat{\varphi}}$.

# $\mathrm{Th}(\mathcal{P}(\omega))$ is model-complete

- Via standard translation, we associate with any $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ an S1S-formula $\widehat{\varphi}(X_1, \ldots, X_n)$ such that, for any $\overline{a} \in \mathcal{P}(\omega)^k$,
$$(\mathcal{P}(\omega), \overline{a}) \models \varphi \iff w_{\overline{a}} \in L_{\widehat{\varphi}}.$$
- By Büchi's theorem, we pick an automaton $\mathcal{A}$ in alphabet $\Sigma_k$ recognizing $L_{\widehat{\varphi}}$.
- Using $(\star)$, we construct an $\mathcal{L}$-term $t_{\mathcal{A}}$ which describes 'successful runs of $\mathcal{A}$'.

# $\mathrm{Th}(\mathcal{P}(\omega))$ is model-complete

- Via standard translation, we associate with any $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ an S1S-formula $\widehat{\varphi}(X_1, \ldots, X_n)$ such that, for any $\overline{a} \in \mathcal{P}(\omega)^k$,
$$(\mathcal{P}(\omega), \overline{a}) \models \varphi \iff w_{\overline{a}} \in L_{\widehat{\varphi}}.$$
- By Büchi's theorem, we pick an automaton $\mathcal{A}$ in alphabet $\Sigma_k$ recognizing $L_{\widehat{\varphi}}$.
- Using $(\star)$, we construct an $\mathcal{L}$-term $t_{\mathcal{A}}$ which describes 'successful runs of $\mathcal{A}$'.
- The term $t_{\mathcal{A}}$ yields an existential $\mathcal{L}$-formula which is equivalent to the formula $\varphi(x_1, \ldots, x_n)$ in the $\mathrm{LTL_I}$-algebra $\mathcal{P}(\omega)$.

# $\mathrm{Th}(\mathcal{P}(\omega))$ is model-complete

- Via standard translation, we associate with any $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ an S1S-formula $\widehat{\varphi}(X_1, \ldots, X_n)$ such that, for any $\overline{a} \in \mathcal{P}(\omega)^k$,
$$(\mathcal{P}(\omega), \overline{a}) \models \varphi \iff w_{\overline{a}} \in L_{\widehat{\varphi}}.$$
- By Büchi's theorem, we pick an automaton $\mathcal{A}$ in alphabet $\Sigma_k$ recognizing $L_{\widehat{\varphi}}$.
- Using $(\star)$, we construct an $\mathcal{L}$-term $t_{\mathcal{A}}$ which describes 'successful runs of $\mathcal{A}$'.
- The term $t_{\mathcal{A}}$ yields an existential $\mathcal{L}$-formula which is equivalent to the formula $\varphi(x_1, \ldots, x_n)$ in the $\mathrm{LTL_I}$-algebra $\mathcal{P}(\omega)$.
- This completes the proof that $T^* = \mathrm{Th}(\mathcal{P}(\omega))$ is model complete.

# Model companion of $\mathrm{LTL_I}$-algebras

Let $\mathcal{L}$ be the first-order language with binary operation $\cup$, unary operations $-$, $\Diamond$, and $\mathbf{X}$, and constant symbols $\emptyset$ and $\mathbf{I}$.

## Theorem

*The theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$ is the model companion of the theory of $\mathrm{LTL_I}$-algebras.*

## Proof outline.

Let $T$ be the theory of $\mathrm{LTL_I}$-algebras and $T^*$ the theory of the $\mathcal{L}$-structure $\mathcal{P}(\omega)$.
Two parts:

1. $T^*$ is a companion of $T$, via duality and filtrations;
2. $T^*$ is model complete, via automata. □

# Conclusion

- We proved that $T^*$ is a model companion of $T$; is it a model completion?

# Conclusion

- We proved that $T^*$ is a model companion of $T$; is it a model completion?
  - (Probably not...)

# Conclusion

- We proved that $T^*$ is a model companion of $T$; is it a model completion?
  - (Probably not...)
- Adapting the proof, one can obtain a version for finite words:

# Conclusion

- We proved that $T^*$ is a model companion of $T$; is it a model completion?
  - (Probably not...)
- Adapting the proof, one can obtain a version for finite words:

**Proposition**

*The theory of the algebra $\mathcal{P}_{fc}(\omega)$ is the model companion of the theory of $\mathrm{LTL_I}$-algebras satisfying the condition $\Box\Diamond x \leq \Diamond\Box x$.*

# Conclusion

- We proved that $T^*$ is a model companion of $T$; is it a model completion?
  - (Probably not...)
- Adapting the proof, one can obtain a version for finite words:

**Proposition**

*The theory of the algebra $\mathcal{P}_{fc}(\omega)$ is the model companion of the theory of $\mathrm{LTL_I}$-algebras satisfying the condition $\Box\Diamond x \leq \Diamond\Box x$.*

- These results are just a first step, many questions remain:

# Conclusion

- We proved that $T^*$ is a model companion of $T$; is it a model completion?
  - (Probably not...)
- Adapting the proof, one can obtain a version for finite words:

**Proposition**

*The theory of the algebra $\mathcal{P}_{fc}(\omega)$ is the model companion of the theory of $\mathrm{LTL_I}$-algebras satisfying the condition $\Box\Diamond x \leq \Diamond\Box x$.*

- These results are just a first step, many questions remain:
  - Can a similar result be obtained for other structures, such as finite trees, infinite trees, ...?

# Conclusion

- We proved that $T^*$ is a model companion of $T$; is it a model completion?
  - (Probably not...)
- Adapting the proof, one can obtain a version for finite words:

**Proposition**

*The theory of the algebra $\mathcal{P}_{fc}(\omega)$ is the model companion of the theory of $\mathrm{LTL_I}$-algebras satisfying the condition $\Box\Diamond x \leq \Diamond\Box x$.*

- These results are just a first step, many questions remain:
  - Can a similar result be obtained for other structures, such as finite trees, infinite trees, ...?
  - What is the general mechanism underlying such results?

# Conclusion

- We proved that $T^*$ is a model companion of $T$; is it a model completion?
  - (Probably not...)
- Adapting the proof, one can obtain a version for finite words:

**Proposition**

*The theory of the algebra $\mathcal{P}_{fc}(\omega)$ is the model companion of the theory of $\mathrm{LTL_I}$-algebras satisfying the condition $\Box\Diamond x \leq \Diamond\Box x$.*

- These results are just a first step, many questions remain:
  - Can a similar result be obtained for other structures, such as finite trees, infinite trees, ...?
  - What is the general mechanism underlying such results?
  - Can these results be used to give (alternative) axiomatizations of monadic second order logics?