Sahlqvist theory for regular modal logics

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Correspondence via Duality

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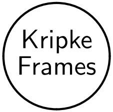

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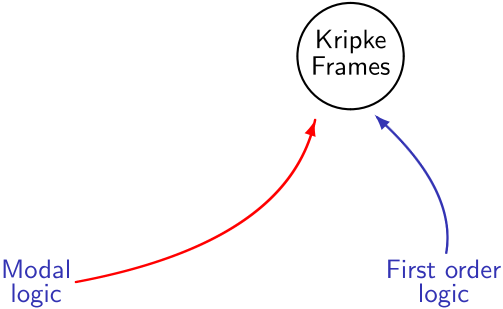

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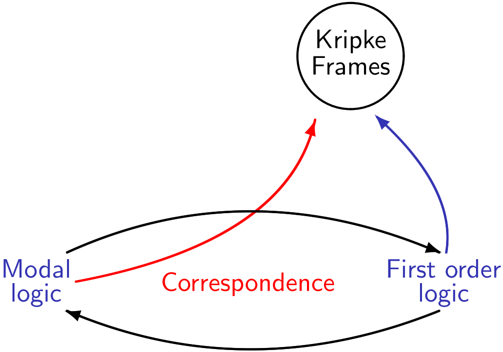
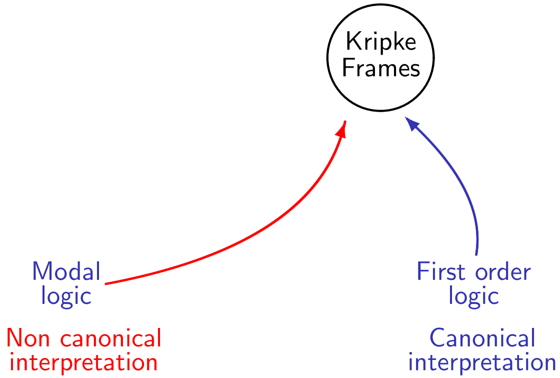
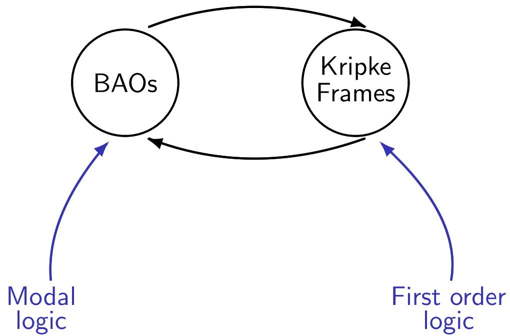


Correspondence via Duality

An asymmetry:

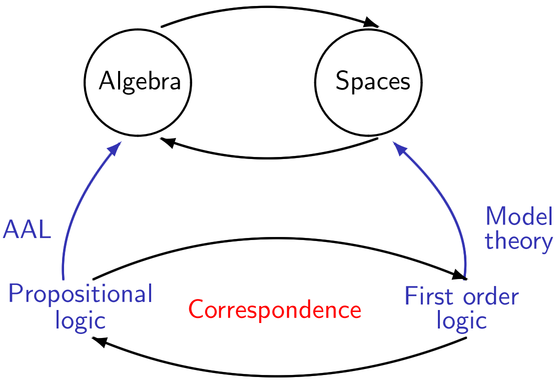


Symmetry re-established via duality:



Correspondence via Duality

Correspondence available not just for normal modal logic:



Hybrid logics
[CR15]

DLE-logics
[CP12, CPS]

Mu-calculi
[CFPS15, CGP14, CC15]

Substructural logics
[CP15]

Regular DLE-logics
Kripke frames with
impossible worlds
[PSZ15a]

Display calculi
[GMPTZ]

Jónsson-style vs
Sambin-style canonicity
[PSZ15b]

Canonicity via
pseudo-correspondence
[CPSZ]

Finite lattices and
monotone ML
[FPS15]

Regular modal logics

Regular modal logics are required to contain the axiom $\Diamond A \leftrightarrow \neg\Box\neg A$ and be closed under the following rule:

$$\text{(RR)}\ \frac{A \wedge B \vdash C}{\Box A \wedge \Box B \vdash \Box C}$$

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No necessitation rule

$$\text{(N)}\ \frac{\vdash A}{\vdash \Box A}$$

Epistemic and deontic logic (Lemmon '57)

If $\Box$ interpreted as "scientific but not logical necessity" or "obligation", then (N) is **not plausible**:
Lemmon:

> *"nothing is a scientific law or a moral obligation as a matter of logic".*

Lemmon's systems E1-5 and D1-5

(1) $\Box(p \to q) \to \Box(\Box p \to \Box q)$	(2) $\Box p \to p$
(1') $\Box(p \to q) \to (\Box p \to \Box q)$	(2') $\Box p \to \Diamond p$
(4) $\Box p \to \Box\Box p$	(5) $\neg\Box p \to \Box\neg\Box p$

E1:	PC+...+(1')+(2)	D1:	PC+...+(1')+(2')
E2:	PC+...+(1')+(2)	D2:	PC+...+(1')+(2')
E3:	PC+...+(1)+(2)	D3:	PC+...+(D)+(1)+(2')
E4:	E2+(4)	D4:	D2+...+(4)
E5:	E2+(5)	D5:	D2+...+(5)

Relational and algebraic semantics

Kripke frame with impossible worlds:

$\mathcal{F} = (W, S, N)$ where:
– $W \neq \varnothing$;
– $S \subseteq N \times W$;
– $N \subseteq W$ is the set of **normal worlds**.
Interpretation of formulas is standard except if $w \in W \setminus N$, for every formula φ,

$$w \not\Vdash \Box\varphi \text{ and } w \Vdash \Diamond\varphi.$$

Regular Boolean algebras expansions:

$\mathbb{B} = (B, \top, \bot, \neg, \wedge, \vee, f)$, such that:
– $(B, \top, \bot, \neg, \wedge, \vee)$ is a Boolean algebra;
– $f : B \to B$ is **additive**:

$$f(a \vee b) = f(a) \vee f(b) \text{ for all } a, b \in B.$$

f is not necessarily normal, i.e. $f(\bot) = \bot$ not true in general.

Discrete duality

For any Kripke frame with impossible worlds $\mathbb{F} = (W, S, N)$, the *complex algebra* associated with $\mathbb{F}$ is $\mathbb{F}^+ := (\mathcal{P}(W), f_S)$ where $f_S : \mathcal{P}(W) \to \mathcal{P}(W)$ is defined by the following assignment:

$$X \mapsto \{w \in W \mid w \notin N \text{ or } wSv \text{ for some } v \in X\} = N^c \cup S^{-1}[X].$$

For every perfect r-BAE $\mathbb{A} = (\mathbb{B}, f)$, the *atom structure with impossible worlds* associated with $\mathbb{A}$ is $\mathbb{A}_+ := (At(\mathbb{A}), S_f, N)$, where $At(\mathbb{A})$ is the collection of atoms of $\mathbb{A}$, $N := \{x \in At(\mathbb{A}) \mid x \nleq f(\bot)\}$ and for all $x, y \in At(\mathbb{A})$ such that $x \nleq f(\bot)$,

$$xS_f y \quad \text{iff} \quad x \leq f(y).$$

Proposition

For every Kripke frame with impossible worlds $\mathbb{F}$ and every perfect r-BAE $\mathbb{A}$,

$$(\mathbb{F}^+)_+ \cong \mathbb{F} \quad \text{and} \quad (\mathbb{A}_+)^+ \cong \mathbb{A}.$$

Normalization of additive maps

With each additive map $f : \mathbb{A} \to \mathbb{B}$ between perfect r-BAEs, we may associate its *normalization*, that is a map

$$\Diamond_f u = \bigvee \{ j \in J^\infty(\mathbb{B}) \mid j \leq f(i) \text{ for some } i \in J^\infty(\mathbb{A}) \text{ such that } i \leq u \}.$$

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By definition, the normalization of f is completely join-preserving. Since perfect lattices are complete, this implies that the normalization is an adjoints, i.e., there exists a map $\blacksquare_f : \mathbb{B} \to \mathbb{A}$ such that for every $u \in \mathbb{A}$ and $v \in \mathbb{B}$,

$$\Diamond_f u \leq v \quad \text{iff} \quad u \leq \blacksquare_f v.$$

Algebraic and Algorithmic Sahlqvist Theory

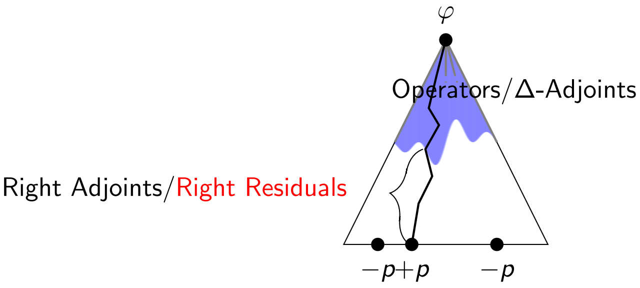
Sahlqvist theory

sufficient **syntactic** conditions on modal formulas:

- to have a first order correspondent;
- to be canonical.

Sahlqvist theory extended:

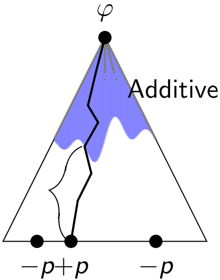
- to (normal) logics on a weaker propositional base (BDL, lattices);
- to more general shapes than Sahlqvist (inductive, recursive).



Our Results

- Algorithmic correspondence theory for regular modal logics on a BDL base;
- Syntactic identification of r-inductive and r-Sahlqvist inequalities;
- Success of Algorithm ALBA^r on r-inductive inequalities;
- Jónsson-style canonicity for r-Sahlqvist inequalities.

Multiplicative in the product



Examples: Completeness w.r.t. Elemantary Classes

The following axioms in Lemmon's system are r-Sahlqvist, and hence canonical.
(2) $\Box p \rightarrow p$ (2') $\Box p \rightarrow \Diamond p$ (4) $\Box p \rightarrow \Box\Box p$ (5) $\neg\Box p \rightarrow \Box\neg\Box p$.

$$
\begin{array}{lll}
 & \forall p(\Box p \leq p) & \\
\text{iff} & \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Box p \ \& \ p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] & \text{(first appr.)} \\
\text{iff} & \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Box\top \ \& \ \blacklozenge\mathbf{i} \leq p \ \& \ p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] & \text{(}\Box\text{-adjunction)} \\
\text{iff} & \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Box\top \ \& \ \blacklozenge\mathbf{i} \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] & \text{(Ackermann rule)} \\
\text{iff} & \forall \mathbf{i}[\mathbf{i} \leq \Box\top \Rightarrow \mathbf{i} \leq \blacklozenge\mathbf{i}] & \\
\text{iff} & \forall x(Nx \rightarrow Rxx). &
\end{array}
$$

Elementary frame condition	**First-order formula**
Normality	$\forall x Nx$
Closure under normality	$\forall x \forall y (Nx \wedge Rxy \rightarrow Ny)$
Pre-normal reflexivity	$\forall x (Nx \rightarrow Rxx)$
Pre-normal transitivity	$\forall x \forall y \forall z (Nx \wedge Ny \wedge Rxy \wedge Ryz \rightarrow Rxz)$
Pre-normal euclideanness	$\forall x \forall y \forall z (Nx \wedge Ny \wedge Rxy \wedge Rxz \rightarrow Ryz)$

Table : Elementary frame conditions

Modal axiom	**Elementary frame condition**
$\Box p \rightarrow p$	Pre-normal reflexivity
$\Box p \rightarrow \Box\Box p$	Pre-normal transitivity and closure under normality
$\neg\Box p \rightarrow \Box\neg\Box p$	Normality and pre-normal euclideanness
$\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$	$\top$
$\Box(p \rightarrow q) \rightarrow \Box(\Box p \rightarrow \Box q)$	Pre-normal transitivity

Table : Lemmon's modal axioms and their elementary frame conditions