Sahlqvist theory for regular modal logics

Alessandra Palmigiano\textsuperscript{1} \quad Sumit Sourabh\textsuperscript{2} \quad Zhiguang Zhao\textsuperscript{1}

\textsuperscript{1}TBM, TU Delft
\textsuperscript{2}ILLC, UvA

TACL 2015
Correspondence via Duality

Correspondence theory arises semantically: Kripke Frames

Modal logic First order logic

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Correspondence an asymmetry: Non canonical interpretation Canonical interpretation

Symmetry re-established via duality: BAOs

Correspondence available not just for normal modal logic: Propositional logic Algebra Spaces Model theory

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- Model theory

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Unified correspondence

Hybrid logics
[CR15]

DLE-logics
[CP12, CPS]

Mu-calculi
[CFPS15, CGP14, CC15]

Substructural logics
[CP15]

Display calculi
[GMPTZ]

Regular DLE-logics
Kripke frames with impossible worlds
[PSZ15a]

Jónsson-style vs Sambin-style canonicity
[PSZ15b]

Finite lattices and monotone ML
[FPS15]

Canonicity via pseudo-correspondence
[CPSZ]
Regular modal logics are required to contain the axiom $\Diamond A \leftrightarrow \neg \Box \neg A$ and be closed under the following rule:

\[(RR) \quad \frac{A \land B \vdash C}{\Box A \land \Box B \vdash \Box C}\]
Regular modal logics are required to contain the axiom $\Diamond A \leftrightarrow \neg \Box \neg A$ and be closed under the following rule:

\[
\text{(RR)} \quad \frac{A \land B \vdash C}{\Box A \land \Box B \vdash \Box C}
\]

No necessitation rule

\[
\text{(N)} \quad \frac{\vdash A}{\vdash \Box A}
\]

Epistemic and deontic logic (Lemmon ’57)

If $\Box$ interpreted as “scientific but not logical necessity” or “obligation”, then (N) is not plausible:

Lemmon:

“nothing is a scientific law or a moral obligation as a matter of logic”.

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Lemmon’s systems E1-5 and D1-5

<table>
<thead>
<tr>
<th></th>
<th>E1:</th>
<th>D1:</th>
</tr>
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<tbody>
<tr>
<td>E1:</td>
<td>PC+...+(1')++(2)</td>
<td>PC+...+(1')++(2')</td>
</tr>
<tr>
<td>E2:</td>
<td>PC+...+(1')++(2)</td>
<td>PC+...+(1')++(2')</td>
</tr>
<tr>
<td>E3:</td>
<td>PC+...+(1)+(2)</td>
<td>PC+...+(D)+(1)+(2')</td>
</tr>
<tr>
<td>E4:</td>
<td>E2+(4)</td>
<td>D2+...+(4)</td>
</tr>
<tr>
<td>E5:</td>
<td>E2+(5)</td>
<td>D2+...+(5)</td>
</tr>
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</table>

\[ (1) \Box(p \rightarrow q) \rightarrow \Box(\Box p \rightarrow \Box q) \]  
\[ (1') \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q) \]  
\[ (2) \Box p \rightarrow p \]  
\[ (2') \Box p \rightarrow \Diamond p \]  
\[ (4) \Box p \rightarrow \Box \Box p \]  
\[ (5) \neg \Box p \rightarrow \Box \neg \Box p \]
Kripke frame with impossible worlds:

\[ \mathcal{F} = (W, S, N) \] where:
- \( W \neq \emptyset \);
- \( S \subseteq N \times W \);
- \( N \subseteq W \) is the set of normal worlds.

Interpretation of formulas is standard except if \( w \in W \setminus N \), for every formula \( \varphi \),

\[ w \not\models \Box \varphi \text{ and } w \models \Diamond \varphi. \]

Regular Boolean algebras expansions:

\[ \mathbb{B} = (B, \top, \bot, \neg, \land, \lor, f), \] such that:
- \( (B, \top, \bot, \neg, \land, \lor) \) is a Boolean algebra;
- \( f : B \to B \) is additive:

\[ f(a \lor b) = f(a) \lor f(b) \text{ for all } a, b \in B. \]

\( f \) is not necessarily normal, i.e. \( f(\bot) = \bot \) not true in general.
Discrete duality

For any Kripke frame with impossible worlds $F = (W, S, N)$, the complex algebra associated with $F$ is $F^+ := (\mathcal{P}(W), f_S)$ where $f_S : \mathcal{P}(W) \rightarrow \mathcal{P}(W)$ is defined by the following assignment:

$$X \mapsto \{w \in W \mid w \notin N \text{ or } wSv \text{ for some } v \in X\} = N^c \cup S^{-1}[X].$$

For every perfect r-BAE $A = (B, f)$, the atom structure with impossible worlds associated with $A$ is $A^+ := (At(A), S_f, N)$, where $At(A)$ is the collection of atoms of $A$, $N := \{x \in At(A) \mid x \notin f(\bot)\}$ and for all $x, y \in At(A)$ such that $x \notin f(\bot)$,

$$xS_f y \iff x \leq f(y).$$

**Proposition**

For every Kripke frame with impossible worlds $F$ and every perfect r-BAE $A$,

$$(F^+)_+ \cong F \quad \text{and} \quad (A_+)^+ \cong A.$$
With each additive map $f : \mathbb{A} \to \mathbb{B}$ between perfect r-BAEs, we may associate its normalization, that is a map

$$\Diamond_f u = \bigvee \{ j \in J^\infty(\mathbb{B}) \mid j \leq f(i) \text{ for some } i \in J^\infty(\mathbb{A}) \text{ such that } i \leq u \}.$$
Normalization of additive maps

With each additive map $f : A \to B$ between perfect $r$-BAEs, we may associate its \textit{normalization}, that is a map

$$\Diamond_f u = \bigvee \{ j \in J^\infty(B) \mid j \leq f(i) \text{ for some } i \in J^\infty(A) \text{ such that } i \leq u \}.$$ 

By definition, the normalization of $f$ is completely join-preserving. Since perfect lattices are complete, this implies that the normalization is an adjoints, i.e., there exists a map $\Box_f : B \to A$ such that for every $u \in A$ and $v \in B$,

$$\Diamond_f u \leq v \iff u \leq \Box_f v.$$
Sahlqvist theory

sufficient **syntactic** conditions on modal formulas:
- to have a first order correspondent;
- to be canonical.

Sahlqvist theory extended:
- to (normal) logics on a weaker propositional base (BDL, lattices);
- to more general shapes than Sahlqvist (**inductive, recursive**).
Our Results

- Algorithmic correspondence theory for regular modal logics on a BDL base;
- Syntactic identification of r-inductive and r-Sahlqvist inequalities;
- Success of Algorithm ALBA^r on r-inductive inequalities;
- Jónsson-style canonicity for r-Sahlqvist inequalities.

\[ \varphi \]

Additive

Multiplicative in the product
The following axioms in Lemmon’s system are r-Sahlqvist, and hence canonical.

(2) $\Box p \rightarrow p$    (2’) $\Diamond p \rightarrow \Diamond p$    (4) $\Box p \rightarrow \Box \Box p$    (5) $\neg \Box p \rightarrow \Box \neg \Box p$.

\[
\forall p (\Box p \leq p)
\]

iff \[
\forall p \forall i \forall m[(i \leq \Box p \land p \leq m) \Rightarrow i \leq m]
\]

(first appr.)

iff \[
\forall p \forall i \forall m[(i \leq \Box \top \land \Diamond i \leq p \land p \leq m) \Rightarrow i \leq m]
\]

($\Box$-adjunction)

iff \[
\forall i \forall m[(i \leq \Box \top \land \Diamond i \leq m) \Rightarrow i \leq m]
\]

(Ackermann rule)

iff \[
\forall i [i \leq \Box \top \Rightarrow i \leq \Diamond i]
\]

iff \[
\forall x (Nx \rightarrow Rxx).
\]
<table>
<thead>
<tr>
<th>Elementary frame condition</th>
<th>First-order formula</th>
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<tr>
<td>Normality</td>
<td>$\forall x Nx$</td>
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<tr>
<td>Closure under normality</td>
<td>$\forall x\forall y (Nx \land Rxy \rightarrow Ny)$</td>
</tr>
<tr>
<td>Pre-normal reflexivity</td>
<td>$\forall x (Nx \rightarrow Rxx)$</td>
</tr>
<tr>
<td>Pre-normal transitivity</td>
<td>$\forall x\forall y\forall z (Nx \land Ny \land Rxy \land Ryz \rightarrow Rxz)$</td>
</tr>
<tr>
<td>Pre-normal euclideanness</td>
<td>$\forall x\forall y\forall z (Nx \land Ny \land Rxy \land Rxz \rightarrow Ryz)$</td>
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Table: Elementary frame conditions

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<td>Pre-normal reflexivity</td>
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<td>$\Box p \rightarrow \Box \Box p$</td>
<td>Pre-normal transitivity and closure under normality</td>
</tr>
<tr>
<td>$\lnot \Box p \rightarrow \Box \lnot \Box p$</td>
<td>Normality and pre-normal euclideanness</td>
</tr>
<tr>
<td>$\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$</td>
<td>$\top$</td>
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Table: Lemmon’s modal axioms and their elementary frame conditions