On strong standard completeness of MTL_*^Q expansions

Amanda Vidal
Joint work with Francesc Esteva and Lluís Godo

Universitat de Barcelona

ARTIFICIAL INTELLIGENCE RESEARCH INSTITUTE - IIIA
CSIC

June 21, 2015

Contents

Contents

The completeness problem

- Logic: Hilbert-style axiomatic system / algebraic semantics.

The completeness problem

- Logic: Hilbert-style axiomatic system / algebraic semantics.
- Many-valued logics: *MTL* and its expansions.

The completeness problem

- Logic: Hilbert-style axiomatic system / algebraic semantics.
- Many-valued logics: *MTL* and its expansions.

- Weaker forms: standard completeness $\mathcal{SC}$ ($\emptyset \vdash_{\mathcal{L}} \varphi \Longleftrightarrow \emptyset \models_{[\mathbf{0},\mathbf{1}]_*} \varphi$), finite standard completeness $\mathcal{FSC}$ ($\gamma_1, \ldots, \gamma_n \vdash_{\mathcal{L}} \varphi \Longleftrightarrow \gamma_1, \ldots, \gamma_n \models_{[\mathbf{0},\mathbf{1}]_*} \varphi$)
- Strong completeness wrt classes of algebras (for any Γ, $\Gamma \vdash_{\mathcal{L}} \varphi \Longleftrightarrow \Gamma \models_{\mathbb{K}} \varphi$.)
- Strong standard completeness $\mathcal{SSC}$ ($\Gamma \vdash_{\mathcal{L}} \varphi \iff \Gamma \models_{[\mathbf{0},\mathbf{1}]_*} \varphi$)
- Strong standard canonical completeness $\mathcal{SSCC}$: language with rational constants & strong completeness wrt. a canonical standard algebra $[\mathbf{0},\mathbf{1}]_*^{\mathbf{Q}}$ (where $\overline{c}^{[\mathbf{0},\mathbf{1}]_*} = c$).

Contents

What is known: general results

- *MTL* strongly complete wrt the class of standard algebras based on left-continuous t-norms
- *BL* is finitely complete wrt the class of standard algebras based on continuous t-norms.

What is known: general results

- *MTL* strongly complete wrt the class of standard algebras based on left-continuous t-norms
- *BL* is finitely complete wrt the class of standard algebras based on continuous t-norms.

standard completeness cases

- Łukasiewicz and Product logics: finite standard completeness (wrt $[\mathbf{0},\mathbf{1}]_{Ł}$ / $[\mathbf{0},\mathbf{1}]_{\Pi}$)

What is known: general results

- *MTL* strongly complete wrt the class of standard algebras based on left-continuous t-norms
- *BL* is finitely complete wrt the class of standard algebras based on continuous t-norms.

standard completeness cases

- Łukasiewicz and Product logics: finite standard completeness (wrt $[\mathbf{0},\mathbf{1}]_{Ł}$ / $[\mathbf{0},\mathbf{1}]_{\Pi}$)
- *BL*: finite standard completeness (wrt a particular standard BL-algebra $[\mathbf{0},\mathbf{1}]_{BL}$)

What is known: general results

- *MTL* strongly complete wrt the class of standard algebras based on left-continuous t-norms
- *BL* is finitely complete wrt the class of standard algebras based on continuous t-norms.

standard completeness cases

- Łukasiewicz and Product logics: finite standard completeness (wrt $[\mathbf{0},\mathbf{1}]_{Ł}$ / $[\mathbf{0},\mathbf{1}]_{\Pi}$)
- *BL*: finite standard completeness (wrt a particular standard BL-algebra $[\mathbf{0},\mathbf{1}]_{BL}$)
- Gödel logic: strong standard complete (the only BL expansion with this property!).

What is known: strong completeness

- Montagna [2006] gives a method to get strong standard axiomatizations of the BL-extensions (wrt all the continuous t-norm based standard algebras) using:
  - Storage operator $*$ (with its corresponding axioms/rules)
  - An infinitary rule

$$\mathrm{R}: \frac{\{\chi \vee (\varphi \to \psi^n)\} \text{ for all n}}{\chi \vee (\varphi \to \psi^*)}$$

What is known: strong completeness

- Montagna [2006] gives a method to get strong standard axiomatizations of the BL-extensions (wrt all the continuous t-norm based standard algebras) using:
  - Storage operator $*$ (with its corresponding axioms/rules)
  - An infinitary rule

$$\mathrm{R}: \frac{\{\chi \vee (\varphi \to \psi^n)\} \text{ for all n}}{\chi \vee (\varphi \to \psi^*)}$$

From here:

1. Ł+ Δ axiomatic system + R is strongly standard complete (wrt $[\mathbf{0},\mathbf{1}]_{\text{Ł}}$)
2. Π + Δ axiomatic system + R is strongly standard complete (wrt $[\mathbf{0},\mathbf{1}]_{\Pi}$)

What is known: Pavelka completeness

Language with truth constants : definition of new type of standard completeness, between finite and strong ones.

What is known: Pavelka completeness

Language with truth constants : definition of new type of standard completeness, between finite and strong ones.

$$
\begin{aligned}
|\varphi|_{\Gamma}^{\mathcal{L}} &= \sup\{c : \Gamma \vdash_{\mathcal{L}} \overline{c} \to \varphi\} \\
\| \varphi \|_{\Gamma}^{\mathcal{L}} &= \inf\{e(\varphi) \text{for } e \text{evaluation} : e[\Gamma] = \{1\}\}
\end{aligned}
$$

What is known: Pavelka completeness

Language with truth constants : definition of new type of standard completeness, between finite and strong ones.

$$
\begin{aligned}
|\varphi|^{\mathcal{L}}_{\Gamma} &= \sup\{c : \ \Gamma \vdash_{\mathcal{L}} \overline{c} \to \varphi\} \\
\| \varphi \|^{\mathcal{L}}_{\Gamma} &= \inf\{e(\varphi) \text{for } e \text{evaluation} : e[\Gamma] = \{1\}\}
\end{aligned}
$$

$\mathcal{L}$ is **Pavelka complete** when $|\varphi|^{\mathcal{L}}_{\Gamma} = \| \varphi \|^{\mathcal{L}}_{\Gamma}$ for all formulas.
$\mathcal{L}$ $\mathcal{SSCC}$ $\Rightarrow$ $\mathcal{L}$ Pavelka complete $\Rightarrow$ $\mathcal{L}$ $\mathcal{FSCC}$.

What is known: Pavelka completeness

Language with truth constants : definition of new type of standard completeness, between finite and strong ones.

$$\begin{aligned} |\varphi|^{\mathcal{L}}_{\Gamma} &= \sup\{c : \ \Gamma \vdash_{\mathcal{L}} \overline{c} \to \varphi\} \\ \| \varphi \|^{\mathcal{L}}_{\Gamma} &= \inf\{e(\varphi) \text{for } e \text{evaluation} : e[\Gamma] = \{1\}\} \end{aligned}$$

$\mathcal{L}$ is **Pavelka complete** when $|\varphi|^{\mathcal{L}}_{\Gamma} = \| \varphi \|^{\mathcal{L}}_{\Gamma}$ for all formulas.
$\mathcal{L}\ \mathcal{SSCC} \Rightarrow \mathcal{L}$ Pavelka complete $\Rightarrow \mathcal{L}\ \mathcal{FSCC}$.
Łukasiewicz logic with rational truth constants has an finitary axiomatic system Pavelka complete: Ł+ book-keeping axioms

$$\overline{c} \& \overline{d} \leftrightarrow \overline{c *_{Ł} d} \qquad \overline{c} \to \overline{d} \leftrightarrow \overline{c \Rightarrow_{Ł} d}$$

(but this is not $\mathcal{SSCC}$)

What is known: Pavelka completeness

Language with truth constants : definition of new type of standard completeness, between finite and strong ones.

$$\begin{aligned} |\varphi|^{\mathcal{L}}_{\Gamma} &= \sup\{c : \ \Gamma \vdash_{\mathcal{L}} \overline{c} \to \varphi\} \\ \| \varphi \|^{\mathcal{L}}_{\Gamma} &= \inf\{e(\varphi) \text{for eevaluation} : e[\Gamma] = \{1\}\} \end{aligned}$$

$\mathcal{L}$ is **Pavelka complete** when $|\varphi|^{\mathcal{L}}_{\Gamma} = \| \varphi \|^{\mathcal{L}}_{\Gamma}$ for all formulas.
$\mathcal{L}$ $\mathcal{SSCC}$ $\Rightarrow$ $\mathcal{L}$ Pavelka complete $\Rightarrow$ $\mathcal{L}$ $\mathcal{FSCC}$.
Łukasiewicz logic with rational truth constants has an finitary axiomatic system Pavelka complete: Ł+ book-keeping axioms

$$\overline{c}\&\overline{d} \leftrightarrow \overline{c *_{Ł} d} \qquad \overline{c} \to \overline{d} \leftrightarrow \overline{c \Rightarrow_{Ł} d}$$

(but this is not $\mathcal{SSCC}$)
this cannot be done with finitary rules if some connective is non-continuous!

What is known: Pavelka completeness II

cf. Prop. 17, [Cintula, 2015]

Let **A** be an expansion of an standard *MTL* algebra with a non-continuous connective, and $\mathcal{L}_{\mathbf{A}}$ an axiomatic system for **A**. Then no finitary rational expansion of $\mathcal{L}_{\mathbf{A}}$ enjoys the Pavelka-style completeness.

What is known: Pavelka completeness II

cf. Prop. 17, [Cintula, 2015]

Let **A** be an expansion of an standard *MTL* algebra with a non-continuous connective, and $\mathcal{L}_{\mathbf{A}}$ an axiomatic system for **A**. Then no finitary rational expansion of $\mathcal{L}_{\mathbf{A}}$ enjoys the Pavelka-style completeness.

- **Infinitary Proof** (of φ from Γ): tree with finite depth where
  - root is φ;
  - leafs are axioms or belong to Γ;
  - for each node θ with descendants $\Theta = \{\theta_i\}$, there is an inference rule $\frac{\Sigma}{\psi}$ and a substitution σ with $\sigma(\psi) = \theta$ and $\sigma(\Sigma) = \Theta$.

What is known: Pavelka completeness III

Cintula and Esteva et. al. propose infinitary rules to manage the discontinuity points of the operations.

What is known: Pavelka completeness III

Cintula and Esteva et. al. propose infinitary rules to manage the discontinuity points of the operations.

- Π logic + Δ op.: $\frac{\{\varphi \to \overline{c}\}_{c \in (0,1)_Q}}{\neg\varphi}$ $\quad$ $\frac{\{\overline{c} \to \varphi\}_{c \in (0,1)_Q}}{\varphi}$

What is known: Pavelka completeness III

Cintula and Esteva et. al. propose infinitary rules to manage the discontinuity points of the operations.

- Π logic + Δ op.: $\frac{\{\varphi \to \overline{c}\}_{c \in (0,1)_Q}}{\neg\varphi}$ $\frac{\{\overline{c} \to \varphi\}_{c \in (0,1)_Q}}{\varphi}$
- General approach for standard algebra of a left-continuous t-norm + canonical constants + extra operations monotonic component-wise: $\mathcal{AS}$ validates a certain infinitary rule for each discontinuity point & is seminilinear $\Longrightarrow$ $\mathcal{AS}$ is Pavelka complete.

What is known: Pavelka completeness III

Cintula and Esteva et. al. propose infinitary rules to manage the discontinuity points of the operations.

- Π logic + Δ op.: $\frac{\{\varphi \to \overline{c}\}_{c\in(0,1)_Q}}{\neg\varphi}$ $\quad$ $\frac{\{\overline{c} \to \varphi\}_{c\in(0,1)_Q}}{\varphi}$
- General approach for standard algebra of a left-continuous t-norm + canonical constants + extra operations monotonic component-wise: $\mathcal{AS}$ validates a certain infinitary rule for each discontinuity point & is seminilinear $\Longrightarrow$ $\mathcal{AS}$ is Pavelka complete.
  - * large variety of additional operations managed
  - * many cases: uncountable infinitary rules
  - * how to know when is a logic with infinitary rules semilinear?..

Contents

Starting point: Δ, constants and semilinearity

$\mathcal{C}_* =$ the countable subalgebra of $[0,1]_*$ gen. by $[0,1] \cap Q$.
$[\mathbf{0},\mathbf{1}]_*^{\mathcal{C}}$ is the standar algebra of $*$ with $\mathcal{C}_*$ canonically interpreted.

Definition

$MTL_\Delta^{\mathcal{C}}$ is the extension of MTL_Δ with the book-keeping axioms for the elements of $\mathcal{C}_*$ over the operations $*$, $\Rightarrow_*$ and Δ.

Starting point: Δ, constants and semilinearity

$\mathcal{C}_* =$ the countable subalgebra of $[0,1]_*$ gen. by $[0,1] \cap Q$.
$[\mathbf{0},\mathbf{1}]^{\mathcal{C}}_*$ is the standar algebra of $*$ with $\mathcal{C}_*$ canonically interpreted.

Definition

$MTL^{\mathcal{C}}_{\Delta}$ is the extension of MTL_{Δ} with the book-keeping axioms for the elements of $\mathcal{C}_*$ over the operations $*$, $\Rightarrow_*$ and Δ.

Semilinearity Lemma

Let $\mathcal{L}$ be an implicative logic expanding[a] $MTL^{\mathcal{C}}_{\Delta}$, such that there is a finite number of infinitary inference rules, and all the rules of $\mathcal{L}$ are closed under $\vee$. Then $\mathcal{L}$ is complete wrt the linear algebras of its eq. algebraic semantics.

[a] with axioms/rules or new operations

The density rule

Idea: discontinuity points could be treated *all at the same time* (i.e., with a unique rule)

The density rule

Idea: discontinuity points could be treated *all at the same time* (i.e., with a unique rule)

- Takeuti and Titani's density (sequents) rule from first order

$$\frac{\Gamma \vdash \chi \vee (\varphi \to x) \vee (x \to \psi)}{\Gamma \vdash \chi \vee (\varphi \to \psi)}$$

where x is propositional variable not in $\Gamma \cup \{\chi, \varphi, \psi\}$.

The density rule

Idea: discontinuity points could be treated *all at the same time* (i.e., with a unique rule)

- Takeuti and Titani's density (sequents) rule from first order

$$\frac{\Gamma \vdash \chi \vee (\varphi \to x) \vee (x \to \psi)}{\Gamma \vdash \chi \vee (\varphi \to \psi)}$$

  where x is propositional variable not in $\Gamma \cup \{\chi, \varphi, \psi\}$.
- Can be translated to an infinitary Hilbert's rule based on constants

Density rule

$$\mathrm{D}^{\infty}: \frac{\{(\varphi \to \overline{c}) \vee (\overline{c} \to \psi)\}_{c \in \mathcal{C}_*}}{(\varphi \to \psi)}$$

note: D^{∞} is closed under $\vee$

An axiomatic system $\mathcal{SSCC}$ for $[\mathbf{0},\mathbf{1}]_*^{\mathcal{C}}$

Definition

MTL_*^{∞} is the extension of $MTL_{\Delta}^{\mathcal{C}}$ with D^{∞}.

An axiomatic system $\mathcal{SSCC}$ for $[\mathbf{0},\mathbf{1}]_*^{\mathcal{C}}$

Definition

MTL_*^{∞} is the extension of $MTL_{\Delta}^{\mathcal{C}}$ with D^{∞}.

MTL_*^{∞} is strongly complete wrt linearly ordered algebras of the class (where the axioms and rules from MTL_*^{∞} hold).

An axiomatic system $\mathcal{SSCC}$ for $[\mathbf{0}, \mathbf{1}]^{\mathcal{C}}_{*}$

Definition

MTL^{∞}_{*} is the extension of $MTL^{\mathcal{C}}_{\Delta}$ with D$^{\infty}$.

MTL^{∞}_{*} is strongly complete wrt linearly ordered algebras of the class (where the axioms and rules from MTL^{∞}_{*} hold).

Lemma: density of constants

If $\mathbf{A}$ is a linearly ordered MTL^{∞}_{*}-algebra, then for any $a < b \in A$ there is $c \in \mathcal{C}_{*}$ such that $a < \overline{c}^{\mathbf{A}} < b$.

Strong standard canonical completeness

Density of constants wrt the elements of linearly ordered algebras limits cardinality of these and allows to prove that $\sigma : A \to [0,1]$ with $\sigma(a) = \sup\{c \in \mathcal{C}_* : \ \overline{c}^{\mathbf{A}} \leq a\} = \inf\{c \in \mathcal{C}_* : \ a \leq \overline{c}^{\mathbf{A}}\}$ is an embedding from **A** into the canonical standard algebra of $*$.

Strong standard canonical completeness

Density of constants wrt the elements of linearly ordered algebras limits cardinality of these and allows to prove that $\sigma : A \to [0,1]$ with $\sigma(a) = \sup\{c \in \mathcal{C}_* : \ \overline{c}^{\mathbf{A}} \leq a\} = \inf\{c \in \mathcal{C}_* : \ a \leq \overline{c}^{\mathbf{A}}\}$ is an embedding from **A** into the canonical standard algebra of $*$.

Completeness

For any set of formulas $\Gamma \cup \{\varphi\}$

$$\Gamma \vdash_{MTL_*^{\infty}} \varphi \iff \Gamma \models_{[\mathbf{0},\mathbf{1}]_*^{\mathcal{C}}} \varphi$$

Contents

Expansions: representable operations

- We can also expand MTL^{∞} with new operations. Which ones?

Expansions: representable operations

- We can also expand MTL^∞ with new operations. Which ones?
- intuitively: **Representable operations** are such whose images can be reached as limits of the images on the rationals from some direction (so the $\vee D^\infty$ rule applies).

Expansions: representable operations

- We can also expand MTL^{∞} with new operations. Which ones?
- intuitively: **Representable operations** are such whose images can be reached as limits of the images on the rationals from some direction (so the $\vee D^{\infty}$ rule applies).
- Our formal suggestions: operations whose universe can be divided in regions left or right continuous and monotone increasing or decreasing component wise (regularity conditions).

Expansions: representable operations

- We can also expand MTL^{∞} with new operations. Which ones?
- intuitively: **Representable operations** are such whose images can be reached as limits of the images on the rationals from some direction (so the $\vee D^{\infty}$ rule applies).
- Our formal suggestions: operations whose universe can be divided in regions left or right continuous and monotone increasing or decreasing component wise (regularity conditions).
- σ must be an embedding for the new operations: regions shall be expressable in the logic, and new rules managing the regularity conditions shall be added to the axiomatic system. In order to easily get semilinearity, we require a finite amount of regions.

Representable operations: formalization

Definition

$\star : [0,1]^n \to [0,1]$ **has a simplifiable universe** when there is $k \in \omega$ and $\{U_i\}_{i \leq k}$, (U_i are its regions) such that

1. $\bigcup_{i \leq k} U_i = [0,1]^n$, and for $i \leq k$, $U_i = U_i^1 \times ..., U_i^n$ with U_i^j being a closed interval of $[0,1]$.
2. For $i \leq k$, $\star$ is component-wise continuous in U_i and component-wise monotonic in *the interior* of U_i;
3. For each $(x_1, ..., x_n) \in [0,1]^n$, either it is a tuple of rationals or there exists U_j such that for each $1 \leq i \leq n$, $x_i \in U_j^i$ and $a < x_i$ if $\star$ is left-continuous in that region/component and $a > x_i$ otherwise.

Representable operations: formalization

Definition

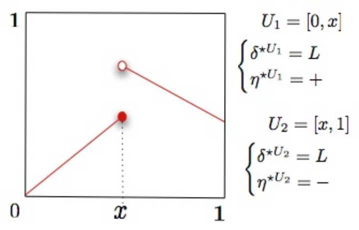
$\star : [0,1]^n \to [0,1]$ **has a simplifiable universe** when there is $k \in \omega$ and $\{U_i\}_{i \leq k}$, (U_i are its regions) such that

1. $\bigcup_{i \leq k} U_i = [0,1]^n$, and for $i \leq k$, $U_i = U_i^1 \times ..., U_i^n$ with U_i^j being a closed interval of $[0,1]$.
2. For $i \leq k$, $\star$ is component-wise continuous in U_i and component-wise monotonic in *the interior* of U_i;
3. For each $(x_1, ..., x_n) \in [0,1]^n$, either it is a tuple of rationals or there exists U_j such that for each $1 \leq i \leq n$, $x_i \in U_j^i$ and $a < x_i$ if $\star$ is left-continuous in that region/component and $a > x_i$ otherwise.

$\star : [0,1]^n \to [0,1]$ is **representable** if it has a simplifiable universe.

Representable functions: examples

example

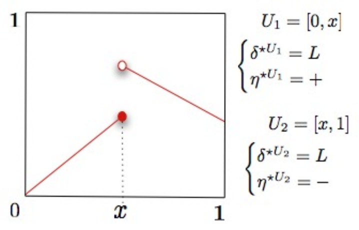


$$\clubsuit(x) :=$$

$$\begin{cases} x & \text{if } b \leq 0.5 \\ 1.2 - x & \text{if } x > 0.5 \end{cases}$$

Representable functions: examples

example



$$♣(x) := \begin{cases} x & \text{if } b \leq 0.5 \\ 1.2 - x & \text{if } x > 0.5 \end{cases}$$

Functions that are not representable are, for instance

- those with punctual discontinuities in non-rational points
- those whose regions cannot be expressed in the logic...

Axiomatic system I

- OP: finite set of representable operations.
- The new language has a symbol $\overline{\star}$ for each $\star \in OP$,
- $\mathcal{C}_*$ (the new constants of the language) are the (countable) subalgebra generated by the rationals with all the operations considered.$[\mathbf{0},\mathbf{1}]_*^{\mathcal{C}}(\mathbf{OP})$ is the corresopnding s
- Some rules will be added...

Axiomatic system I

- OP: finite set of representable operations.
- The new language has a symbol $\overline{\star}$ for each $\star \in OP$,
- $\mathcal{C}_*$ (the new constants of the language) are the (countable) subalgebra generated by the rationals with all the operations considered.$[\mathbf{0}, \mathbf{1}]^{\mathcal{C}}_{*}(\mathbf{OP})$ is the corresopnding s
- Some rules will be added...

Raisowa-implicativity:

$$(\vee\text{CONG}^{\star})\frac{\gamma \vee \{\varphi_1 \leftrightarrow \psi_1, ..., \varphi_n \leftrightarrow \psi_n\}}{\gamma \vee (\overline{\star}(\varphi_1, ..., \varphi_n) \rightarrow \overline{\star}(\psi_1, ..., \psi_n))}$$

Axiomatic system I

- OP: finite set of representable operations.
- The new language has a symbol $\overline{\star}$ for each $\star \in OP$,
- $\mathcal{C}_*$ (the new constants of the language) are the (countable) subalgebra generated by the rationals with all the operations considered.$[\mathbf{0},\mathbf{1}]_*^{\mathcal{C}}(\mathbf{OP})$ is the corresopnding s
- Some rules will be added...

Raisowa-implicativity:

$$(\vee\text{CONG}^{\star})\frac{\gamma \vee \{\varphi_1 \leftrightarrow \psi_1, ..., \varphi_n \leftrightarrow \psi_n\}}{\gamma \vee (\overline{\star}(\varphi_1, ..., \varphi_n) \to \overline{\star}(\psi_1, ..., \psi_n))}$$

$$(\vee\text{CONG}^{\clubsuit})\frac{\gamma \vee (\varphi \leftrightarrow \psi)}{\gamma \vee (\overline{\clubsuit}\varphi \to \overline{\clubsuit}\psi)}$$

Axiomatic system II

Some notation previous to the regularity rules:

Axiomatic system II

Some notation previous to the regularity rules:

- regularity characteristics of $\star$ in a region U in component i:

monotonicity: $\eta_i^{\star U}$	continuity: $\delta_i^{\star U}$
increasing: $\eta_i^{\star U} = +$	left: $\delta_i^{\star U} = L$
decreasing: $\eta_i^{\star U} = -$	right: $\delta_i^{\star U} = R$

Axiomatic system II

Some notation previous to the regularity rules:

- regularity characteristics of $\star$ in a region U in component i:

monotonicity: $\eta_i^{\star U}$	continuity: $\delta_i^{\star U}$
increasing: $\eta_i^{\star U} = +$	left: $\delta_i^{\star U} = L$
decreasing: $\eta_i^{\star U} = -$	right: $\delta_i^{\star U} = R$

$$\eta^{\clubsuit[0,0.5]} = +, \eta^{\clubsuit[0.5,1]} = -,$$
$$\delta^{\clubsuit[0,0.5]} = L, \delta^{\clubsuit[0.5,1]} = L$$

Axiomatic system II

Some notation previous to the regularity rules:

- regularity characteristics of $\star$ in a region U in component i:

monotonicity: $\eta_i^{\star U}$	continuity: $\delta_i^{\star U}$
increasing: $\eta_i^{\star U} = +$	left: $\delta_i^{\star U} = L$
decreasing: $\eta_i^{\star U} = -$	right: $\delta_i^{\star U} = R$

$$\eta^{\clubsuit[0,0.5]} = +, \eta^{\clubsuit[0.5,1]} = -,$$
$$\delta^{\clubsuit[0,0.5]} = L, \delta^{\clubsuit[0.5,1]} = L$$

- $impl(+, \varphi, \psi) = impl(L, \varphi, \psi) = \varphi \to \psi$
  $impl(-, \varphi, \psi) = impl(R, \varphi, \psi) = \psi \to \varphi$

Axiomatic system II

Some notation previous to the regularity rules:

- regularity characteristics of $\star$ in a region U in component i:

monotonicity: $\eta_i^{\star U}$	continuity: $\delta_i^{\star U}$
increasing: $\eta_i^{\star U} = +$	left: $\delta_i^{\star U} = L$
decreasing: $\eta_i^{\star U} = -$	right: $\delta_i^{\star U} = R$

$$\eta^{\clubsuit[0,0.5]} = +, \eta^{\clubsuit[0.5,1]} = -,$$
$$\delta^{\clubsuit[0,0.5]} = L, \delta^{\clubsuit[0.5,1]} = L$$

- $impl(+, \varphi, \psi) = impl(L, \varphi, \psi) = \varphi \to \psi$
  $impl(-, \varphi, \psi) = impl(R, \varphi, \psi) = \psi \to \varphi$
- Extreme uncontrolled (under continuity) points:

$$\begin{aligned} \chi_{U^i}(x) &= x \leftrightarrow \overline{extr_{U^i}} \\ extr_{U^i} &= \begin{cases} \min U^i & \text{if } \delta^{\star U^i} = L \\ \max U^i & \text{if } \delta^{\star U^i} = R \end{cases} \end{aligned}$$

Monotonicity rules

For each region U of the simplified universe of $\star$, and each component i:

$$(\vee \mathrm{M}_{\mathrm{i}}^{\star \mathrm{U}}) \ \frac{\gamma \vee \{(\varphi_1, ..., \varphi_i, ..., \varphi_n) \in U, x \in U^i, \\ impl(\eta_i^{\star U}, x, \psi), impl(\eta_i^{\star U}, \psi, \varphi_i)\}}{\gamma \vee \chi_{U^i}(x) \vee (\overline{\star}(\varphi_1, ..., \psi, ..., \varphi_n) \rightarrow \overline{\star}(\varphi_1, ..., \varphi_i, ..., \varphi_n))}$$

where x does not appear in any other formula

Monotonicity rules

For each region U of the simplified universe of $\star$, and each component i:

$$(\vee \mathrm{M}_{\mathrm{i}}^{\star \mathrm{U}})\ \frac{\gamma \vee \{(\varphi_1, ..., \varphi_i, ..., \varphi_n) \in U, x \in U^i, \; impl(\eta_i^{\star U}, x, \psi), impl(\eta_i^{\star U}, \psi, \varphi_i)\}}{\gamma \vee \chi_{U^i}(x) \vee (\overline{\star}(\varphi_1, ..., \psi, ..., \varphi_n) \to \overline{\star}(\varphi_1, ..., \varphi_i, ..., \varphi_n))}$$

where x does not appear in any other formula

$$(\vee \mathrm{M}^{\clubsuit[0,0.5]})\frac{\gamma \vee \{\varphi \to \overline{0.5}, x \to \psi, \psi \to \varphi\}}{\gamma \vee (x \leftrightarrow \overline{0}) \vee (\overline{\clubsuit}(\psi) \to \overline{\clubsuit}(\varphi)}$$

$$(\vee \mathrm{M}^{\clubsuit[0.5,1]})\frac{\gamma \vee \{\overline{0.5} \to \varphi, \psi \to x, \varphi \to \psi\}}{\gamma \vee (x \leftrightarrow \overline{0.5}) \vee (\overline{\clubsuit}(\psi) \to \overline{\clubsuit}(\varphi)}$$

Continuity rules

If $\star$ is **left-continuous and increasing** in U^i ($\delta_i^{\star U} = L, \eta_i^{\star U} = +$) or **right-continuous and decreasing** ($\delta_i^{\star U} = R, \eta_i^{\star U} = -$):

$$
(\mathrm{C}_\mathrm{i}^{\star \mathrm{U}})\ \frac{\begin{array}{c}\gamma \vee \{(\varphi_1, \dots, \varphi_n) \in U, x \to \overline{\star}(\varphi_1, \dots, \varphi_n), \\ \{\chi_{U^i}(\overline{d}) \vee impl(\delta_i^{\star U}, x_i, \overline{d}) \vee \\ \overline{\star}(\varphi_1, \dots, \overline{d}, \dots, \varphi_n) \to x\}_{d \in U^i \cap \mathcal{C}_*}\}\end{array}}{\gamma \vee (\overline{\star}(\varphi_1, \dots, \varphi_i, \dots, \varphi_n) \to x)}
$$

Continuity rules

If $\star$ is **left-continuous and increasing** in U^i $(\delta_i^{\star U} = L, \eta_i^{\star U} = +)$ or **right-continuous and decreasing** $(\delta_i^{\star U} = R, \eta_i^{\star U} = -)$:

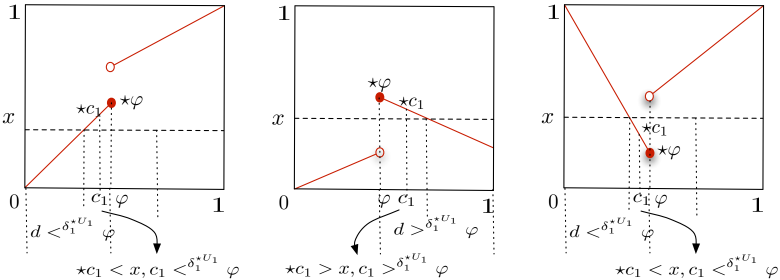
$$(\mathrm{C_i^{\star U}})\ \frac{\gamma \vee \{(\varphi_1,...,\varphi_n) \in U, x \rightarrow \overline{\star}(\varphi_1,...,\varphi_n), \{\chi_{U^i}(\overline{d}) \vee impl(\delta_i^{\star U}, x_i, \overline{d}) \vee \overline{\star}(\varphi_1,...,\overline{d},...,\varphi_n) \rightarrow x\}_{d \in U^i \cap \mathcal{C}_*}\}}{\gamma \vee (\overline{\star}(\varphi_1,...,\varphi_i,...,\varphi_n) \rightarrow x)}$$

If $\star$ is **left-continuous and decreasing** in U^i $(\delta_i^{\star U} = L, \eta_i^{\star U} = -)$ or **right-continuous and increasing** $(\delta_i^{\star U} = R, \eta_i^{\star U} = +)$:

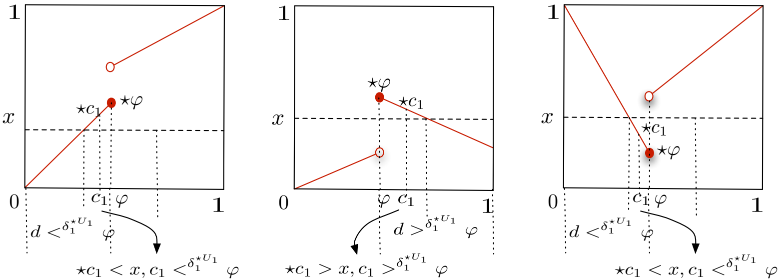
$$(\mathrm{C_i^{\star U}})\ \frac{\gamma \vee \{(\varphi_1,...,\varphi_n) \in U, \overline{\star}(\varphi_1,...,\varphi_n) \rightarrow x, \{\chi_{U^i}(\overline{d}) \vee impl(\delta_i^{\star U}, x_i, \overline{d}) \vee x \rightarrow \overline{\star}(\varphi_1,...,\overline{d},...,\varphi_n)\}_{d \in U^i \cap \mathcal{C}_*}\}}{\gamma \vee (x \rightarrow \overline{\star}(\varphi_1,...,\varphi_i,...,\varphi_n))}$$

where x does not appear in other formulas.

What was that?



What was that?



$$(\vee C^{\clubsuit[0,0.5]})\ \frac{\gamma \vee \{\varphi \to \overline{0.5}, x \to \overline{\clubsuit}(\varphi) \qquad \{(\overline{c} \leftrightarrow \overline{0}) \vee (\varphi \to \overline{c}) \vee (\overline{\clubsuit}(\overline{c}) \to x)\}_{c\in[0,0.5]\cap\mathcal{C}_*}\}}{\gamma \vee (\overline{\clubsuit}(\varphi) \to x)}$$

$$(\vee C^{\clubsuit[0.5,1]})\ \frac{\gamma \vee \{\overline{0.5} \to \varphi, \overline{\clubsuit}(\varphi) \to x \qquad \{(\overline{c} \leftrightarrow \overline{0.5}) \vee (\varphi \to \overline{c}) \vee (x \to \overline{\clubsuit}(\overline{c})\}_{c\in[0.5,1]\cap\mathcal{C}_*}\}}{\gamma \vee (x \to \overline{\clubsuit}(\varphi))}$$

Completeness again

Definition

$MTL^{\infty}_{*}(OP) = MTL^{\infty}_{*}$ over the language with $\overline{\star}$ and $\mathcal{C}_{*}$ plus:

- book-keeping axioms (for $\mathcal{C}_{*}$) for each $\star$ in OP.
- congruence rule of each $\star$
- regularity rules for each $\star$

Completeness again

Definition

$MTL_*^\infty(OP) = MTL_*^\infty$ over the language with $\barwedge$ and $\mathcal{C}_*$ plus:

- book-keeping axioms (for $\mathcal{C}_*$) for each $\star$ in OP.
- congruence rule of each $\star$
- regularity rules for each $\star$

- Previous rules hold the premises of the Semilinearity Lemma, so $MTL_*^\infty(OP)$ is strongly complete wrt linearly ordered algebras from its class.
- $\sigma(x) = \sup\{c \in \mathcal{C}_* : \overline{c}^{\mathbf{A}} \leq x\} = \inf\{c \in \mathcal{C}_* : \overline{c}^{\mathbf{A}} \geq x\}$ is an embedding from each linearly ordered $\mathbf{A}$ into $[\mathbf{0},\mathbf{1}]_*^{\mathcal{C}}(\mathbf{OP})$.

Completeness again

Definition

$MTL^{\infty}_{*}(OP) = MTL^{\infty}_{*}$ over the language with $\overline{\ast}$ and $\mathcal{C}_{*}$ plus:

- book-keeping axioms (for $\mathcal{C}_{*}$) for each $\star$ in OP.
- congruence rule of each $\star$
- regularity rules for each $\star$

- Previous rules hold the premises of the Semilinearity Lemma, so $MTL^{\infty}_{*}(OP)$ is strongly complete wrt linearly ordered algebras from its class.
- $\sigma(x) = \sup\{c \in \mathcal{C}_{*} : \overline{c}^{\mathbf{A}} \leq x\} = \inf\{c \in \mathcal{C}_{*} : \overline{c}^{\mathbf{A}} \geq x\}$ is an embedding from each linearly ordered $\mathbf{A}$ into $[\mathbf{0},\mathbf{1}]^{\mathcal{C}}_{*}(\mathbf{OP})$.

Theorem

$$\Gamma \vdash_{MTL^{\infty}_{*}(OP)} \varphi \iff \Gamma \models_{[\mathbf{0},\mathbf{1}]^{\mathcal{C}}_{*}(\mathbf{OP})} \varphi$$

Contents

Some remarks

- It is not possible, in general, to give a finite axiomatization for Pavelka complete logics (and so, for strong standard canonical complete logics).

Some remarks

- It is not possible, in general, to give a finite axiomatization for Pavelka complete logics (and so, for strong standard canonical complete logics).
- The first-order density rule can be adapted to obtain a rule determining the density of the constant elements over the linearly-ordered algebras.

Some remarks

- It is not possible, in general, to give a finite axiomatization for Pavelka complete logics (and so, for strong standard canonical complete logics).
- The first-order density rule can be adapted to obtain a rule determining the density of the constant elements over the linearly-ordered algebras.
- The Δ operation allows a uniform proof of strong completeness wrt chains.

Some remarks

- It is not possible, in general, to give a finite axiomatization for Pavelka complete logics (and so, for strong standard canonical complete logics).
- The first-order density rule can be adapted to obtain a rule determining the density of the constant elements over the linearly-ordered algebras.
- The Δ operation allows a uniform proof of strong completeness wrt chains.
- This is enough for the *MTL* logics. We can treat some of its expansions too, when the operations behave somehow *regularly*.

Some remarks

- It is not possible, in general, to give a finite axiomatization for Pavelka complete logics (and so, for strong standard canonical complete logics).
- The first-order density rule can be adapted to obtain a rule determining the density of the constant elements over the linearly-ordered algebras.
- The Δ operation allows a uniform proof of strong completeness wrt chains.
- This is enough for the *MTL* logics. We can treat some of its expansions too, when the operations behave somehow *regularly*.

- Is Δ truly necessary?

Some remarks

- It is not possible, in general, to give a finite axiomatization for Pavelka complete logics (and so, for strong standard canonical complete logics).
- The first-order density rule can be adapted to obtain a rule determining the density of the constant elements over the linearly-ordered algebras.
- The Δ operation allows a uniform proof of strong completeness wrt chains.
- This is enough for the MTL logics. We can treat some of its expansions too, when the operations behave somehow *regularly*.

- Is Δ truly necessary?
- Can we treat a wider family of operations?

Thank you!

Thank you!

F. Montagna.
Notes on strong completeness in Łukasiewicz, product and BL logics and in their first-order extensions.
In *Algebraic and Proof-theoretic Aspects of Non-classical Logics*, pages 247–274, 2006.

P. Cintula.
A note on axiomatizations of pavelka-style complete fuzzy logics.
Fuzzy Sets and Systems, (0), 2014.

F. Esteva, L. Godo, and C. Noguera.
On completeness results for predicate Łukasiewicz, product, Gödel and nilpotent minimum logics expanded with truth-constants.
Mathware and Soft Computing, 14(3):223–246, 2007.